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OPTIMAL PORTFOLIO STRATEGIES WITH STOCHASTIC WAGE INCOME AND INFLATION: THE CASE OF A DEFINED CONTRIBUTION PENSION PLAN

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Optimal Portfolio Strategies with Stochastic Wage Income and Inflation: The Case of a Defined Contribution Pension Plan[*]

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Abstract

We consider a stochastic model for a defined-contribution pension fund in continuous time. In particular, we focus on the portfolio problem of a fund manager who wants to maximize the expected utility of his terminal wealth in a complete financial market with stochastic interest rate. The fund manager must cope with a set of stochastic investment opportunities and two background risks: the salary risk and the inflation risk. We use the stochastic dynamic programming approach. We show that the presence of the inflation risk can solve some problems linked to the use of the stochastic dynamic programming technique, and namely to the stochastic partial differential equation deriving from it. We find a closed form solution to the asset allocation problem, without specifying any functional form for the coefficients of the diffusion processes involved in the problem. Finally, the derivation of a closed form solution allows us to analyse in detail the behaviour of the optimal portfolio with respect to salary and inflation.

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1 Introduction

There are two extremely different ways to manage a pension fund. On the one hand, we find defined-benefit plans (hereafter DB), where benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. On the other hand, there are defined-contribution plans (hereafter DC), where contributions are fixed and benefits depend on the returns on fund's portfolio. However, DC plans allow contributors to know, at each time, the value of their retirement accounts. Historically, fund managers have mainly proposed DB plans, which are definitely preferred by workers. In fact, in the case of DB plans, the associated financial risks are supported by the plan sponsor rather than the individual member of the plan. Nowadays, most of the proposed pension plans are based on DC schemes involving a considerable transfer of risks to workers. Accordingly, DC pension funds provide contributors with a service of saving management, without guaranteeing any minimum performance. As we have already highlighted, only contributions are fixed in advance, while the final retirement account fundamentally depends on the administrative and financial skill of the fund managers. Therefore, an efficient financial management is essential to gain contributors' trust.

The goal of the fund manager is to invest the accumulated wealth in order to optimize the expected value of a suitable terminal utility function. The classical dynamic optimization model, initially proposed by Merton (1971), assumes a market structure with constant interest rate. In the case of pension funds, the optimal asset-allocation problem involves quite a long period, generally from 20 to 40 years. It follows that the assumption of constant interest rates is not fit for our purpose. Moreover, the benefits proposed by DC pension plans often require the specification of the stochastic behavior of other variables, such as salaries. Thus, the fund manager must cope not only with financial risks, but also with background risks, where by "financial risks" we mean the risks involved by the financial market, and by "background risks" we mean all the risks outside the financial market (e.g. salary and inflation).

Merton (1969, 1971, 1990), Duffie (1996), and Karatzas and Shreve (1998) provide general treatment of optimal portfolio choice in continuous-time, without any background risk. Actually, the optimal portfolio problem becomes more and more complex when we allow for background risk. At this regard, it is important to distinguish between two different classes of background risks: the "level" background risk and the "ratio" background risk. The first set of risk affects the amount of wealth which can be invested while the second set of risk affects only the wealth growth rate. In this work, we consider both kinds of risk and, in particular, a scalar dimensional "level" background risk given by the shareholder's salary and a scalar dimensional "ratio" background risk given by the inflation rate. Both processes are stochastic and, as shown in Menoncin (2002), the presence of the inflation risk can solve some problems linked to the use of the stochastic dynamic programming technique, and namely to the stochastic partial differential equation deriving from it.

In this work, we consider the stochastic model for pension fund dynamics in continuous time presented by Battocchio (2001), where we introduce the inflation risk. We assume a financial market with stochastic interest rate and consisting of

three assets: a riskless asset, a stock, and a bond, which can be bought and sold without incurring any transaction costs or restriction on short sales. More precisely, we study an optimal investment problem related to the accumulation phase of a defined-contribution pension fund. We consider the case of a shareholder who, at each period $t \in [0, T]$, contributes a constant proportion of his salary to a personal pension fund. At the time of retirement T, the accumulated pension fund will be converted into an annuity. Similar models have been recently presented by Blake, Cairns, and Dowd (2000), Boulier, Huang, and Taillard (2001) and Deelstra, Grasselli, and Koehl (2001). Especially, Blake et al. (2000) assume a stochastic process for salary including a non-hedgeable risk component and focus on the replacement ratio as the central quantity of interest. Boulier et al. (2001) assume a deterministic process of salary and consider a guarantee on the benefits. Accordingly, they strongly support the real need for a downside protection of contributors who are more directly exposed to the financial risk borne by the pension fund. Also Deelstra et al. (2001) allow for a minimum guarantee in order to minimize the randomness of the retirement account, but they describe the contribution flow through a non-negative, progressive measurable, and square-integrable process. A recent model for a DC pension scheme in discrete time is proposed by Haberman and Vigna (2001). In particular, they study both the "investment risk", that is the risk of incurring a poor investment performance during the accumulation phase of the fund, and the "annuity risk", that is the risk of purchasing an annuity at retirement in a particular recessionary economic scenario involving a low conversion rate.

The problem of optimal portfolio choice for a long-term investor in the presence of wage income is also treated by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2002), and Franke, Peterson, and Stapleton (2001). Under a complete market with a constant interest rate, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labor income. Thus, they assume that the income process does not involve a new source of uncertainty. Campbell and Viceira (2002) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyse the impact of the resolution for the labor income uncertainty on portfolio choice. They show how the investor's portfolio startegy changes when his labor income uncertainty is resolved earlier or later in life.

The methodological approach we use to solve the optimal asset-allocation problem of a pension fund is the stochastic dynamic programming. Alternative approaches (see for instance Deelstra et al. (2001), and Lioui and Poncet (2001)) are based on the Cox-Huang methodology (the so called martingale approach), where the resulting partial differential equation is often simpler to solve than the Hamilton-Jacobi-Bellman equation coming from the dynamic programming. We just underline that in this work we are able to reach the same qualitative results as Lioui and Poncet even if they do not consider any inflation risk. In this work, we use the methodology developped in Menoncin (2002). In his model the fund managers do not maximize the expected utility of fund's terminal wealth but the difference beteween this wealth and the performance on a given benchmark. In this work we consider inflation as the benchmark and thus we suppose that fund's managers maximize the expected utility of fund's real terminal wealth.

By following this way, we are able to find a closed form solution to the asset allocation problem, without specifying any functional form for the coefficients of the diffusion processes involved in the problem. We underline that, on the contrary, the exact solutions generally found for the optimal portfolio problem have been based on precise assumptions on the behaviour of the state variables. In particular, in the above mentioned literature, these variables are supposed to follow a Vasicek process (Vasicek, 1977) or a CIR process (Cox, Ingersoll, and Ross, 1985).

The derivation of a closed form solution allows us to analyse in detail the behaviour of the optimal portfolio with respect to salaries and inflation. Furthermore, we show that it is optimal to invest in a combination of three portfolios: a speculative portfolio proportional to the market price of risk of the risky assets through the risk aversion index, an hedging portfolio proportional to the diffusion term of the instantaneous interest rate, and a preference-free hedging portfolio proportional to the diffusion term of the salary process.

The work is organized as follows. Section 2 presents the general framework and exposes the financial market structure, the stochastic processes describing the behaviour of asset prices, the background risks (i.e. salaries and inflation), and the fund wealth. Section 3 presents the main results. The optimal portfolio allocation is computed, and an explicit solution to the dynamic stochastic problem is derived. Section 4 concludes. Some technical details about the diffusion processes presented in the model are exposed in two appendices.

2   The Model

In this section we introduce the market structure of the optimal asset allocation model, we define the stochastic dynamics of the interest rate and asset values, and we present the stochastic processes describing the behaviour of the two background risks: salaries and inflation. The coefficients of all stochastic processes involved in the model are assumed to meet the usual regularity conditions, necessary for having a unique solution to the stochastic differential equations.

We consider a complete and frictionless financial market which is continuosly open over the fixed time interval $[0, T]$, where $T \in \mathbb{R}^+ \setminus \{0\}$ denotes the retirement time of a representative shareholder. The uncertainty involved by the financial market is described by two standard Brownian motions $W^0(t)$, and $W^1(t)$ with $t \in [0, T]$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, $\mathcal{F} = \{\mathcal{F}(t)\}_{t \in [0,T]}$ is the filtration generated by the Brownian motions and $\mathbb{P}$ represents the historical probability measure. The filtration $\mathcal{F}(t)$ can be interpreted as the information set available to the investor at time t. The two Wiener processes $W^0(t)$ and $W^1(t)$ are supposed to be independent. We can impose this constraint without loss of generality. In fact, we can always shift from uncorrelated to correlated Wiener processes (and viceversa) via the Cholesky decomposition of the correlation matrix. A description of the Cholesky decomposition of the correlation matrix is provided in Appendix A.

2.1 The Financial Market

We consider a general one-factor model for the forward interest rate $f(t, \tau)$, whose dynamics is given by:

$$
\begin{aligned}
df(t, \tau) &= \alpha(t, \tau) \, dt + \nu(t, \tau) \, dW^0(t), \quad \tau \in [0, T], \quad (1) \\
f(0, \tau) &= f_0,
\end{aligned}
$$

where we assume $\nu(t, \tau) > 0$.[1] Following Björk (1998), from the forward interest rate we can derive the behaviour of the spot interest rate.

Proposition 1 *If $f(t, \tau)$ satisfies Equation (1), then the short rate satisfies*

$$
dr(t) = a(t)dt + b(t)dW^0(t), \quad (2)
$$

where

$$
\begin{cases}
a(t) = \frac{\partial}{\partial \tau} f(t, \tau)|_{\tau=t} + \alpha(t, t), \\
b(t) = \nu(t, t).
\end{cases}
$$

We assume that the fund manager can invest in three assets: a riskless asset, a bond, and a stock.

The price process $X^0(t, r)$ of the riskless asset is given by:

$$
\begin{aligned}
dX^0(t, r) &= X^0(t, r)r(t)dt, \quad (3) \\
X^0(0) &= 1,
\end{aligned}
$$

where the dynamics of $r(t)$, under the real probability measure $\mathbb{P}$, is defined in Equation (2). The riskless asset can be considered as a bank account, paying the instantaneous interest rate $r(t)$ without any default risk.

The second asset we introduce is a bond rolling over zero coupon bonds with maturity τ, where $\tau \in [0, T]$.

Given the forward interest rate (1), we assume that there exists a market for zero coupon bonds for every value of τ. It is known that a zero coupon bond with maturity τ, called τ-bond, is a contract which guarantees the holder 1 monetary unit (face value) to be paid on the maturity τ. We denote by $X^B(t, \tau)$ the price at time $t \in [0, \tau[$ of a zero coupon bond with maturity τ. Then, we have a market with an infinite number of bonds, where each bond is regarded as a derivative of the underlying riskless asset. Thus, each bond is characterized by the same market price of risk (see for example Björk, 1998). Now, when the market has specified the dynamics of a basic bond price process, say with maturity τ, the market has also indirectly specified the price of risk which is the same for each bond, as we have already noted. Then, the basic τ-bond and the forward interest rate fully determine the price of all bonds. Actually, assuming the existence of an infinite number of zero coupon bonds is quite unrealistic. However, since the forward rate dynamics has only one source of randomness, we only need one zero coupon bond to replicate the other ones. Following Björk (1998), the bond dynamics is defined as follows:

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[1] We assume a strictly positive diffusion term for the forward interest rate, since it leads to a strictly negative volatility term in the bond's dynamics. In fact, when the interest rate increases, then the bond value decreases. Thus, the two diffusion terms must have opposite sign.

Proposition 2 *If $f(t, \tau)$ satisfies Equation (1), then $X^B(t, \tau)$ satisfies*

$$\frac{dX^B(t, \tau)}{X^B(t, \tau)} = \mu_B(t, \tau)dt + \sigma_B(t, \tau)dW^0(t), \tag{4}$$

where

$$\begin{cases} \mu_B(t, \tau) = r(t) - \int_t^\tau \alpha(t, s)\, ds + \frac{1}{2}\left(\int_t^\tau \nu(t, s)\, ds\right)^2, \\ \sigma_B(t, \tau) = -\int_t^\tau \nu(t, s)\, ds. \end{cases} \tag{5}$$

The third asset we consider is a stock. For the sake of simplicity, we introduce in our model only one stock, which can be interpreted as a stock market index. Nevertheless, if we allow for a complete market with a finite number of stocks, no further difficulties are added to the model because the only source of troubles is the market incompleteness.

The dynamics of the stock price $X^1(t, r)$ is given by:

$$\begin{aligned} dX^1(t, r) &= X^1(t, r)\left[\mu_1(t, r)dt + \sigma_{1,0}(t, r)dW^0(t) + \sigma_{1,1}(t, r)dW^1(t)\right], \tag{6} \\ X^1(0) &= X_0^1, \end{aligned}$$

where $\sigma_{1,0}(t, r) \neq 0$ and $\sigma_{1,1}(t, r) \neq 0$.

The diffusion matrix for the considered financial market is given by:

$$\Sigma(t, \tau, r) \equiv \begin{bmatrix} \sigma_{0,1}(t, r) & \sigma_{1,1}(t, r) \\ \sigma_B(t, \tau) & 0 \end{bmatrix}.$$

As $\sigma_{1,1}(t, r) \neq 0$ and $\sigma_B(t, \tau) \neq 0$, it follows that

$$\det \Sigma(t, \tau, r) = -\sigma_{1,1}(t, r)\sigma_B(t, \tau) \neq 0,$$

thus, consistently with the assumption of complete market, the diffusion matrix $\Sigma(t, \tau, r)$ is invertible.

2.2 The Defined-Contribution Process

The introduction in the optimal portfolio problem of no-capital income causes several computational difficulties, although the underlying methodological approach is the same as that one used for the no-wage income case. In general, because of the presence of background risks directly affecting the wealth level (e.g. salaries), the solution of the partial differential equation (PDE) characterizing the stochastic optimal control problem becomes harder and harder to compute. However, since our goal is to analyse the optimal portfolio strategies for a DC pension fund during the accumulation phase, then we cannot overlook the leading role of the salary process.

Merton (1971), in his seminal stochastic dynamic optimization framework, examines the effects of introducing a deterministic wage income in the consumption-portfolio problem. In the more recent literature, Boulier et al. (2001), and Deelstra et al. (2001) provide some models for DC pension funds in continuous time involving deterministic salaries. Blake et al. (2000) consider a model for DC pension funds

where salaries are modeled through a stochastic process including a non-hedgeable component. Haberman and Vigna (2001) provide a model for DC pension funds in discrete time with a fixed contribution rate. The problem of optimal portfolio choice for a long-term investor in presence of wage income is treated also by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2002), and Franke, Peterson, and Stapleton (2001). Under a complete market with a constant interest rate, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labor income. Thus, they assume that the income process does not involve a new source of uncertainty. On the opposite, we introduce in the defined-contribution process a non-hedgeable risk component. Campbell and Viceira (2002) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyse the impact of labor income uncertainty resolution on portfolio choice. They show how the portfolio startegy of an investor changes when his labor income uncertainty is resolved early or late in life. In particular, they add the labor income to the terminal value of the portfolio investments. In the present work, the income process enters in the wealth process at each time $t \in [0, T]$.

This paper is principally related to the work of Blake et al. (2000) and Battocchio (2002), even if we are able to compute a close form solution to our dynamic problem thanks to the methodology developed in Menoncin (2002).

Indeed, we characterize the salary process through a stochastic differential equation. Accordingly, we will show that the optimal portfolio choice curcially depends on the uncertainty involved by salary. The introduction of stochastic salaries, instead of deterministic, allows us to consider the effects due to the labor income uncertainty, and in particular to its resolution over time.

The dynamic evolution of salaries is given by:

$$\frac{dS(t,r)}{S(t,r)} = \mu_S(t,r)dt + \sigma_{S,0}(t,r)dW^0(t) + \sigma_{S,1}(t,r)dW^1(t) + \sigma_S(t)dW^S(t), \quad (7)$$

where $W^S(t)$ is a one-dimensional standard Brownian motion independent of $W^0(t)$ and $W^1(t)$. As Blake et al. (2000) point out, the assumption of a time-dependent drift term $\mu_S(t,r)$ allows us to incorporate possible age-dependent salary growth. At this regard, it is well known that salaries grow faster at younger ages. This empirical evidence suggests a decreasing function of time for the drift term $\mu_S(t,r)$. The diffusion terms $\sigma_{S,0}(t,r)$ and $\sigma_{S,1}(t,r)$ allow us to model any link between salary growth and returns on bond and stock. We note again that the unique stock we have introduced in our simple model can be always interpreted as the index of the stock market. According to the stochastic salary process modeled by Blake et al. (2000), the term $\sigma_S(t)dW^S(t)$ allows us to incorporate non-hedgeable salary risks, that is risks properly associated with the labor market and not with the financial market.

Now, we assume that each employee puts a constant proportion γ of his salary into the personal pension fund. Then, the defined contribution process is characterized as follows:

$$C(t,r) = \gamma S(t,r),$$

and
$$\frac{dC(t,r)}{C(t,r)} = \frac{dS(t,r)}{S(t,r)},$$
so, in this model, the contribution growth rate equals the wage growth rate.

2.3 The Inflation

Almost all the literature about the optimal portfolio rules does not consider the inflation rate, and the asset prices are specified in nominal terms. In this work, we offer a solution to the optimal asset allocation problem for a DC pension fund. Since the period of time considered in this framework is quite long (from 20 to 40 years), then we must cope also with the inflation risk.

Inflation can be considered as a background risk affecting only the wealth growth rate without altering the amount of wealth that can be invested. Actually, fund managers have to invest the nominal fund, although they are interested in maximizing the growth rate of the real fund. Then, we have to consider two different measures for the same fund. In particular, we call F_N the nominal fund and F the real fund.

For modelling how the real fund behaves, a common used approximation is the following one: the growth rate of the real fund is given by the difference between the nominal fund growth rate and the consumption price growth rate. If we call p the level of consumption prices, then we can write:

$$\frac{dF}{F} \simeq \frac{dF_N}{F_N} - \frac{dp}{p}. \tag{8}$$

This is the so called Fisher equation but it gives a log-approximation of the exact relation which must hold between F_N and F. Actually, the true relation comes from an arbitrage hypothesis. Considering the inflation rate in this framework means to consider a possible arbitrage between the financial and the real market. In fact, the nominal interest rate must compensate the opportunity cost of investing in financial assets. The investor who puts his money in the financial market misses the return he could have obtained from a real investment. If the investor buys today a real good and sells it after one period, he gains the inflation rate. If he buys today a financial asset and sells it after one period, he gains a nominal return. Now, we suppose that a particular market, called "real-financial" market, exists. If this is the case, the corresponding "real-financial" return must be such that the investor is indifferent between the two following opportunities:

1. investing one nominal monetary unit in the financial market and missing the return he could have obtained on the real market;

2. investing one nominal monetary unit in the "real-financial" market.

Accordingly, if we call ϕ the "real-financial" return, ϕ_N the nominal financial return, and π the inflation rate, then the true equation that must hold between the nominal and the real fund is the following one:

$$F\phi = F_N\phi_N - F_N\pi,$$

which means that the return on the real wealth must equate the return on the nominal wealth reduced by the loss due to the increase in the price level. By definition it must be true that:

$$\phi \equiv \frac{dF}{F}, \qquad \phi_N \equiv \frac{dF_N}{F_N}, \qquad \pi \equiv \frac{dp}{p},$$

and so, after substituting in the arbitrage condition, we can write:

$$dF = dF_N - F_N \frac{dp}{p}. \tag{9}$$

After defining this arbitrage condition, we have to make some hypotheses about the behaviour of consumption prices. In particular, we suppose that p follows a diffusion process of the following form:

$$\frac{dp(t)}{p(t)} = \mu_\pi(t,r)\,dt + \sigma_{\pi,0}(t,r)dW^0(t) + \sigma_{\pi,1}(t,r)dW^1(t) + \sigma_\pi(t)\,dW^S,$$

where $\sigma_\pi(t) \neq 0$. Here, we suppose that the risk sources affecting the consumption price growth rate are the same as those affecting the subscribers' wages.

We underline that, after introducing the inflation risk, the riskless asset looses its characteristics, and it is no more able to guarantee a riskless return because it cannot hedge against the inflation risk. It is important to stress that this simple property cannot be found by using the approximation (8), while the exact formula (9) accounts for the change of the riskless asset into a risky asset. This is the reason why, in this framework, the Fisher's approximation is not acceptable.

Actually, after introducing the inflation risk, the market completeness must be defined not only on the two original risky assets (the stock and the bond), but also on a new risky asset, whose drift is given by the riskless rate net of the inflation rate, and whose diffusion term is given by the inflation diffusion.

2.4 The Pension Fund

Let us summarize the whole market structure as follows, where, for the sake of simplicity, we leave out the functional dependences:

$$\begin{cases} dr = \mathrm{a}dt + \mathrm{b}dW^0, \\[2mm] \frac{dX^0}{X^0} = rdt, \\[2mm] \frac{dX^1}{X^1} = \mu_1 dt + \sigma_{1,0}dW^0 + \sigma_{1,1}dW^1, \\[2mm] \frac{dX^B}{X^B} = \mu_B dt + \sigma_B dW^0, \end{cases} \tag{10}$$

$$\frac{dC}{C} = \mu_S dt + \sigma_{S,0}dW^0 + \sigma_{S,1}dW^1 + \sigma_S dW^S,$$
$$\frac{dp}{p} = \mu_\pi dt + \sigma_{\pi,0}dW^0 + \sigma_{\pi,1}dW^1 + \sigma_\pi dW^S.$$

Let $\theta_1(t,r)$, $\theta_B(t,r)$, and $\theta_0(t,r)$ denote the amount of money invested in the two risky assets (i.e. the stock and the bond) and in the riskless asset respectively. Thus, the accumulated nominal wealth at any time $t \in [0,T]$ is given by:

$$F_N(t,r) = \theta_0(t,r) + \theta_1(t,r) + \theta_B(t,r) + C(t,r). \tag{11}$$

The change in the nominal fund F_N is given by the amount of money invested in each asset multiplied by the return on each asset and by the change in the contributions. Thus, after differentiating Equation (11), we obtain that

$$dF_N = \theta_0 \frac{dX^0}{X^0} + \theta_1 \frac{dX^1}{X^1} + \theta_B \frac{dX^B}{X^B} + dC.$$

Now, we use the definition of the real fund dynamics (9) for writing:

$$dF = \theta_0 \frac{dX^0}{X^0} + \theta_1 \frac{dX^1}{X^1} + \theta_B \frac{dX^B}{X^B} + dC - (\theta_0 + \theta_1 + \theta_B + C) \frac{dp}{p},$$

which can be also written as:

$$
\begin{aligned}
dF = {} & \theta_0 \left(\frac{dX^0}{X^0} - \frac{dp}{p} \right) + \theta_1 \left(\frac{dX^1}{X^1} - \frac{dp}{p} \right) + \\
& + \theta_B \left(\frac{dX^B}{X^B} - \frac{dp}{p} \right) + C \left(\frac{dC}{C} - \frac{dp}{p} \right).
\end{aligned}
$$

In this way we can see that the riskless asset includes, with a negative sign, the diffusion term of the inflation process. Hence, it becomes a risky asset. After substituting the differentials from system (10), we obtain that

$$
\begin{aligned}
dF = {} & [\theta_0 (r - \mu_\pi) + \theta_1 (\mu_1 - \mu_\pi) + \theta_B (\mu_B - \mu_\pi) + C (\mu_S - \mu_\pi)] \, dt + \\
& + [-\theta_0 \sigma_{\pi,0} + \theta_1 (\sigma_{1,0} - \sigma_{\pi,0}) + \theta_B (\sigma_B - \sigma_{\pi,0}) + C (\sigma_{S,0} - \sigma_{\pi,0})] \, dW^0 + \\
& + [-\theta_0 \sigma_{\pi,1} + \theta_1 (\sigma_{1,1} - \sigma_{\pi,1}) - \theta_B \sigma_{\pi,1} + C (\sigma_{S,1} - \sigma_{\pi,1})] \, dW^1 + \\
& - [\theta_0 \sigma_\pi + \theta_1 \sigma_\pi + \theta_B \sigma_\pi - C (\sigma_S - \sigma_\pi)] \, dW^S.
\end{aligned}
$$

For the sake of simplicity we define:

$$
\begin{aligned}
\theta &\equiv \begin{bmatrix} \theta_0 & \theta_1 & \theta_B \end{bmatrix}', \\
M &\equiv \begin{bmatrix} (r - \mu_\pi) & (\mu_1 - \mu_\pi) & (\mu_B - \mu_\pi) \end{bmatrix}', \\
\Gamma &\equiv \begin{bmatrix} -\sigma_{\pi,0} & -\sigma_{\pi,1} & -\sigma_\pi \\ \sigma_{1,0} - \sigma_{\pi,0} & \sigma_{1,1} - \sigma_{\pi,1} & -\sigma_\pi \\ \sigma_B - \sigma_{\pi,0} & -\sigma_{\pi,1} & -\sigma_\pi \end{bmatrix}, \\
\Lambda &\equiv \begin{bmatrix} (\sigma_{S,0} - \sigma_{\pi,0}) & (\sigma_{S,1} - \sigma_{\pi,1}) & (\sigma_S - \sigma_\pi) \end{bmatrix}', \\
W &\equiv \begin{bmatrix} W^0 & W^1 & W^S \end{bmatrix}',
\end{aligned}
$$

where the prime denotes transposition. After this simplification, we can write the dynamic behaviour of the real fund in the following way:

$$dF = \left[\theta' M + C (\mu_S - \mu_\pi) \right] dt + \left(\theta' \Gamma + C \Lambda' \right) dW. \tag{12}$$

We underline that the new diffusion matrix for the financial market is given by Γ which must be invertible if we want this market to be complete. In this case, we have:

$$\det(\Gamma) = \sigma_B \sigma_\pi \sigma_{1,1},$$

which is different from zero because $\sigma_{1,1}$ and σ_π are different from zero by hypothesis, while $\sigma_B \neq 0$ by construction (see Equations (5)). Thus, the financial market is complete even after the introduction of the inflation risk. In fact, the inflation increases the number of risk sources, but also the number of risky assets is increased by one, due to the change of the riskless asset into a risky asset.

3 The Optimal Asset Allocation Problem

The goal of the fund manager is to choose a portfolio stategy in order to maximize the expected value of a terminal utility function $K(F(T))$. We assume that the terminal utility K is an increasing and concave function of F. Then, we may formally state the stochastic optimal control problem as follows:

$$\begin{cases} \underset{\theta}{Max} \quad \mathbb{E}_0\left[K(F(T)) \mid F(0) = F_0, r(0) = r_0\right] \\ d\begin{bmatrix} r \\ F \end{bmatrix} = \begin{bmatrix} a \\ \theta'M + C(\mu_S - \mu_\pi) \end{bmatrix}dt + \begin{bmatrix} \delta' \\ \theta'\Gamma + C\Lambda' \end{bmatrix}dW, \end{cases} \tag{13}$$

where $\delta \equiv \begin{bmatrix} b & 0 & 0 \end{bmatrix}'$. The scalar variables F and r represent the two state variables, while the elements of θ represent the three control variables.

The methodology used to solve this optimal control problem is the stochastic dynamic programming. By the theory (e.g. Björk, 1998), we know that the original optimal control problem is equivalent to the problem of finding a solution to a suitable partial differential equation (PDE), known as the Hamilton-Jacobi-Bellman (HJB) equation. Under our assumption, the HJB equation provides a very nice solution to the optimal control problem in which we are considering only Markov processes. We will not describe rigorously the whole theoretical structure of this approach, but we will limit our analysis to the basic steps necessary to specify the HJB equation which characterizes our optimal control problem.

Furthermore, we have written Problem (13) in a way useful for applying the exact solution, exposed in Menoncin (2002), to this kind of framework. Accordingly, we are able to offer a close form solution to the optimal portfolio problem for a defined contribution pension fund.

Let $J(t; F_0, r_0)$ denote the value function of our optimal control problem (13), then it follows that

$$J(t; F_0, r_0) = \mathbb{E}_t\left[K(F(T)) \mid F(0) = F_0, r(0) = r_0\right],$$

where $\mathbb{E}_t$ stands for $\mathbb{E}\left(\cdot|\mathcal{F}(t)\right)$.

The Hamiltonian corresponding to (13) results to be:

$$\begin{aligned} \mathcal{H} &= J_r a + J_F\left[\theta'M + C(\mu_S - \mu_\pi)\right] + \\ &\quad + \frac{1}{2}tr\left\{\begin{bmatrix} \delta' \\ \theta'\Gamma + C\Lambda' \end{bmatrix}\begin{bmatrix} \delta & \Gamma'\theta + C\Lambda \end{bmatrix}\begin{bmatrix} J_{rr} & J_{rF} \\ J_{Fr} & J_{FF} \end{bmatrix}\right\}, \end{aligned}$$

where we denote $J_r \equiv \frac{\partial J}{\partial r}$, $J_F \equiv \frac{\partial J}{\partial F}$, $J_{rr} \equiv \frac{\partial^2 J}{\partial r^2}$, $J_{FF} \equiv \frac{\partial^2 J}{\partial F^2}$, and $J_{rF} = J_{Fr} \equiv \frac{\partial^2 J}{\partial r \partial F}$.

By working out the Hamiltonian, we obtain that:

$$
\begin{aligned}
\mathcal{H} = {} & J_r a + J_F \left[\theta' M + C(\mu_S - \mu_\pi) \right] + \\
& + \frac{1}{2} J_{rr} \delta' \delta + J_{rF} \left(\theta' \Gamma + C\Lambda' \right) \delta + \\
& + \frac{1}{2} J_{FF} \left[\theta' \Gamma \Gamma' \theta + 2C\theta' \Gamma \Lambda + C^2 \Lambda' \Lambda \right].
\end{aligned}
$$

The first order condition gives us the following linear system of three equations and three unknows:

$$
\frac{\partial \mathcal{H}}{\partial \theta} = J_F M + J_{Fr} \Gamma \delta + J_{FF} \left(\Gamma \Gamma' \theta + C\Gamma \Lambda \right) = 0. \tag{14}
$$

We note that J_{FF} must be strictly negative. Indeed, the second order condition holds if the corresponding Hessian matrix, given by

$$
J_{FF} \Gamma \Gamma',
$$

is negative definite. As $\Gamma \Gamma'$ is a variance-covariance matrix, it is a positive definite matrix. Then, the Hessian matrix is a negative definite matrix if and only if $J_{FF} < 0$. Under our assumptions, we can easily show that J_{FF} is effectively strictly negative. In fact, since the terminal utility K is concave in F, the value function J results to be a strictly concave function in F (see for example Stokey and Lucas, 1989). Moreover, the completeness of the market implies that the matrix $\Gamma \Gamma'$ is invertible.

Let $\theta^*(t,r) = \begin{bmatrix} \theta_0^*(t,r) & \theta_1^*(t,r) & \theta_B^*(t,r) \end{bmatrix}'$ denote the vector of optimal amounts of wealth invested in each asset. From Equation (14), we obtain:

$$
\theta^* = -\frac{J_F}{J_{FF}} \left(\Gamma \Gamma' \right)^{-1} M - \frac{J_{Fr}}{J_{FF}} \left(\Gamma' \right)^{-1} \delta - C \left(\Gamma' \right)^{-1} \Lambda, \tag{15}
$$

where $J(t, F, r)$ solves the following partial differential equation:

$$
\begin{cases}
J_t + \mathcal{H}^* = 0, \\
J(T, F(T), r(T)) = K(F(T)),
\end{cases} \tag{16}
$$

called Hamilton-Jacobi-Bellman equation, where $J_t \equiv \frac{\partial J}{\partial t}$, and $\mathcal{H}^*$ denotes the value of the Hamiltonian with respect to the optimal proportions θ^*. The hard work in the stochastic dynamic programming approach consists just in solving the highly nonlinear PDE involved by the optimal control problem. Nevertheless, in this framework we are able to replicate the result obtained by Menoncin (2002) who finds an exact solution for this dynamic optimal stochastic control problem. In the next section we analyse the properties of the optimal portfolio and we derive the exact solution of the dynamic problem.

3.1 The Optimal Portfolio

In this section, we analyse some structural properties of the optimal portfolio $\theta^*(t,r)$. In particular, we show that the optimal portfolio can be interpreted as the sum of three different components in the following way:

$$\theta^* (t,r) = \phi_1 (t,r) \, p_1 (t,r) + \phi_2 (t,r) \, p_2 (t,r) + p_3 (t,r) \, .$$

Comparing this with Equation (15), we find that it is optimal to invest in a suitable combination of three portfolios:

1. a speculative portfolio $\phi_1 p_1$ proportional to the market price of risk corresponding to the three risky assets through the absolute risk aversion index, and defined as follows:

$$\phi_1 p_1 \equiv -\frac{J_F}{J_{FF}} \left(\Gamma\Gamma' \right)^{-1} M,$$

where

$$\phi_1 \equiv -\frac{J_F}{J_{FF}}$$

represents the coefficient of this portfolio component depending on individual preferences;

2. an hedging portfolio $\phi_2 p_2$ proportional to the diffusion term of the interest rate through the cross derivative of the value function J with respect to the fund and the interest rate, and defined as follows:

$$\phi_2 p_2 \equiv -\frac{J_{Fr}}{J_{FF}} \left(\Gamma' \right)^{-1} \delta,$$

where:

$$\phi_2 \equiv -\frac{J_{Fr}}{J_{FF}}$$

represents the coefficient of this portfolio component depending on individual preferences;

3. a preference-free hedging component p_3 proportional to the volatilities of the salary process and inversely related with the asset volatilities, and defined as follows:

$$p_3 \equiv -C \left(\Gamma' \right)^{-1} \Lambda.$$

We underline that p_i, $i = \{1,2,3\}$, are colomn vectors whose dimensions equal the number of assets on the market, while ϕ_j, $j = \{1,2\}$, are scalar values.

We note that p_1 and p_2 are two components depending only on the financial market structure. Accordingly, the investment policies corresponding to p_1 and p_2 do not require the knowledge of the preferences or the endowments of the fund's shareholders. It follows that p_1 and p_2 are the same for all participants. On the other hand, ϕ_1 and ϕ_2 directly depend on the individual preferences. The interpretation of this result in terms of pension fund management is that the optimal portfolio is set up by two purely financial components, p_1 and p_2, common to every shareholder,

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and which must be adjusted on the basis of the individual preferences through ϕ_1 and ϕ_2.

Let us analyse the third component of the optimal portfolio:

$$
p_3 = -C \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \frac{\sigma_{S,0} - \sigma_{\pi,0}}{\sigma_B} +
$$

$$
-C \begin{bmatrix} \sigma_{1,0} - \sigma_B \\ \sigma_B \\ -\sigma_{1,0} \end{bmatrix} \frac{\sigma_{S,1} - \sigma_{\pi,1}}{\sigma_{1,1}\sigma_B} +
$$

$$
-C \begin{bmatrix} -\sigma_{1,0}\sigma_{\pi,1} - \sigma_{1,1}\sigma_B + \sigma_B\sigma_{\pi,1} + \sigma_{\pi,0}\sigma_{1,1} \\ -\sigma_{\pi,1}\sigma_B \\ -\sigma_{\pi,0}\sigma_{1,1} + \sigma_{1,0}\sigma_{\pi,1} \end{bmatrix} \frac{\sigma_S - \sigma_\pi}{\sigma_\pi\sigma_{1,1}\sigma_B}.
$$

As noted, p_3 represents an hedging component of the optimal portfolio θ^* and it is preference-free. It follows that p_3 is completely defined without specifying the functional form of the value function J. We see that an increase in the volatility of salaries with respect to the risk of the stock market ($\sigma_{S,1}$) immediately draws in a fall, both in the proportion invested in the stock, and in the proportion invested in the bond (we recall that $\sigma_B < 0$). Therefore, an increase in $\sigma_{S,1}$ involves an increase in the proportion invested in the riskless asset. On the other hand, a rise in the volatility of salaries with respect to the risk of the interest rate ($\sigma_{S,0}$) does not affect the investment in the stock. In this case, we have a rise in the optimal proportion invested in the bond, and a corresponding decrease in the investment in the riskless asset.

The effect of the volatility of the non-hedgeable term (σ_S) is much more uncertain, but we can easily see that if σ_S increases, then the amount of money invested in the stock also increases.

All the effects we have underlined for the contribution diffusion terms are the same, but with the opposite sign, for the inflation diffusion terms. In this way, if the contribution volatility with respect to the interest rate ($\sigma_{S,0}$) increases, but the inflation corresponding volatility ($\sigma_{\pi,0}$) also increases by the same amount, then the optimal portfolio composition does not change. This property is true for all the volatility terms of contribution and inflation processes. Each element of p_3 is linked with the corresponding element of the diffusion term of salaries net of inflation risk.

It is important to observe that if we assume deterministic salaries, then the preference-free component (p_3) disappears because we have $\sigma_{S,0} = \sigma_{S,1} = \sigma_S = 0$ and all the three components of p_3 equal zero. In fact, in the work by Boulier et al. (2001) deterministic salaries are considered and their optimal portfolio is formed by only two components.

The element p_2 can be written in the following way:

$$p_2 = \begin{bmatrix} -\frac{b}{\sigma_B} \\ 0 \\ \frac{b}{\sigma_B} \end{bmatrix},$$

from which we can see that the stock does not contribute at all to the hedging component (its coefficient equals 0), while the riskless asset and the bond contribute with the same coefficient but with the opposite sign. This coefficient is given by the ratio between the diffusion term of the spot interest rate (b) and the diffusion term of the bond (σ_B). Since there exists a precise relation between these two variables (see Equation (5)), then we can also write:

$$p_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{\nu(t,t)}{\int_t^\tau \nu(t,s)\,ds}.$$

Thus, we can conclude that the preference free part of the optimal portfolio second component does depend only on the volatility of the interest rate. This means that, when on the financial market the interest rate volatility increases, the amount of wealth that must be invested in the riskless asset decreases, and the wealth invested in the bond increases by the same amount.

3.2 The Value Function: An Exact Solution

In the previous section, we have highlighted some interesting properties of the optimal portfolio $\theta^*(t,r)$, without specifing the value function J. In order to precisely state its functional form, we should solve the HJB equation. In general, there is no analytical method to solve an highly nonlinear PDE. Nevertheless, our framework is able to replicate the market structure analysed in Menoncin (2002), where the author is able to find an exact solution to the asset allocation problem.

If we substitute for the optimal asset allocation θ^* into the Hamiltonian, we obtain that

$$\mathcal{H}^* = J_r \mathrm{a} + J_F C \Lambda' \Gamma^{-1} M - J_F C \left(\mu_S - \mu_\pi \right) + \tag{17}$$
$$+ \frac{1}{2} J_{rr} \delta' \delta - \frac{1}{2} \frac{(J_{Fr})^2}{J_{FF}} \delta' \delta - \frac{J_{Fr} J_F}{J_{FF}} \delta' \Gamma^{-1} M +$$
$$- \frac{1}{2} \frac{(J_F)^2}{J_{FF}} M' \left(\Gamma \Gamma' \right)^{-1} M.$$

In the financial literature, since Merton (1969, 1971), the condition of separability in wealth by product represents a common assumption in the attempt to explicitely solve the optimal portfolio problem. Accordingly, we assume that our value function is given by the product of two terms: an increasing and concave function of the wealth F, and an exponential function depending on time and on the interest rate r. Then, the value function J can be written as follows:

$$J(t,r,F) = U(F)e^{h(t,r)}.$$

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Substituting this into the HJB equation (17), we obtain that

$$\begin{cases} J\left(t,r,F\right)h_t + \mathcal{H}^* = 0, \\ h(T,r(T)) = 0, \end{cases}$$

and, after dividing by J, we can write the HJB equation in the following way:

$$\begin{aligned} 0 &= h_t + h_r \mathrm{a} + \frac{U_F}{U}C\left[\Lambda'\Gamma^{-1}M - (\mu_S - \mu_\pi)\right] + \\ &\quad + \frac{1}{2}\left(h_r^2 + h_{rr}\right)\delta'\delta - \frac{1}{2}\frac{(U_F)^2}{U_{FF}U}h_r^2\delta'\delta - \frac{(U_F)^2}{U_{FF}U}h_r\delta'\Gamma^{-1}M + \\ &\quad - \frac{1}{2}\frac{(U_F)^2}{U_{FF}U}M'\left(\Gamma\Gamma'\right)^{-1}M. \end{aligned}$$

In order to have our model consistent with the assumption of separability in wealth, we must impose that $\frac{U_F}{U}$ and $\frac{(U_F)^2}{U_{FF}U}$ are constant with respect to the wealth F. The only function complying with these conditions is the exponential utility function:

$$U\left(F\right) = \beta_1 e^{\beta_2 + \beta_3 F},$$

for which we have:

$$\begin{aligned} \frac{U_F}{U} &= \beta_3, \\ \frac{(U_F)^2}{U_{FF}U} &= 1. \end{aligned}$$

Accordingly, the HJB equation can be written as follows:

$$\begin{aligned} 0 &= h_t + h_r\left[\mathrm{a} - \delta'\Gamma^{-1}M\right] + \frac{1}{2}h_{rr}\delta'\delta + \\ &\quad + \beta_3 C\left[\Lambda'\Gamma^{-1}M - (\mu_S - \mu_\pi)\right] - \frac{1}{2}M'\left(\Gamma\Gamma'\right)^{-1}M. \end{aligned}$$

This kind of partial differential equation can be solved thanks to the Feynman-Kac theorem,[2] and so we can find the functional form of $h\left(r,t\right)$, which is given by:

$$h\left(r,t\right) = \mathbb{E}_t\left[\int_t^T z\left(\widetilde{r}\left(s\right),s\right)ds\right],$$

where:[3]

$$\begin{aligned} d\widetilde{r}\left(s\right) &= \left[\mathrm{a} - \mathrm{b}\frac{\mu_B - \widetilde{r}\left(s\right)}{\sigma_B}\right]ds + \nu dW^0, \\ \widetilde{r}\left(t\right) &= r, \\ z\left(\widetilde{r}\left(t\right),t\right) &= \beta_3 C\left[\Lambda'\Gamma^{-1}M - (\mu_S - \mu_\pi)\right] - \frac{1}{2}M'\left(\Gamma\Gamma'\right)^{-1}M. \end{aligned}$$

---

[2] For a complete exposition of the Feynman-Kac theorem, the reader is referred to Duffie (1996), Björk (1998), and Øksendal (2000).

[3] We underline that, after working out the matrix notation $\delta'\Gamma^{-1}M$, we can write:

$$\delta'\Gamma^{-1}M = \mathrm{b}\frac{\mu_B - r\left(s\right)}{\sigma_B}.$$

Finally, we can write the optimal portfolio in the following way:

$$\theta^* = -\frac{1}{\beta_3}\left(\Gamma\Gamma'\right)^{-1}M - \frac{1}{\beta_3}\left(\Gamma'\right)^{-1}\delta\int_t^T\frac{\partial}{\partial r}\mathbb{E}_t\left[z\left(\widetilde{r}\left(s\right),s\right)ds\right]ds - C\left(\Gamma'\right)^{-1}\Lambda.$$

The only preference parameter entering θ^* is β_3 which is the Arrow-Pratt risk aversion index.

We underline that, under the Faynman-Kac representation theorem, the interest rate is given by the solution to a stochastic differential equation which is different from the original one. In particular, the interest rate giving the exact solution to our problem has the same diffusion term as the original interest rate in Equation (2), while the drift term is different. The new drift term $\widetilde{a}$ is given by the original drift term diminished by the bond Sharpe ratio multiplied by the interest rate diffusion term. Nevertheless, this new drift term can be also written as follows:

$$\begin{aligned}\widetilde{a} &\equiv a - b\frac{\mu_B - \widetilde{r}\left(s\right)}{\sigma_B} = \\ &= a - \frac{b}{\sigma_B}\mu_B + \frac{b}{\sigma_B}\widetilde{r}\left(s\right).\end{aligned}$$

As the terms a and b depend only on time, then the modified interest rate solving the optimal portfolio problem follows an extended Vasicek model, defined in Hull and White (1990).

In the work by Lioui and Poncet (2000) it is shown that, if the market is complete, then the third component of the optimal portfolio is formed only by two parts, even though the number of state variables is arbitrarily large. In particular, the first part is associated with the interest rate risk and the second one with the market price of risk. Even if Lioui and Poncet use the martingale approach, here we underline that we obtain the same qualitative result.

Because the authors do not introduce any contribution process,[4] then we have to put in our framework $C = 0$. Under this hypothesis we can see that the function $h\left(z,t\right)$ is formed only by one term and, more precisely, we have:

$$z\left(\widetilde{r}\left(t\right),t\right) = -\frac{1}{2}M'\left(\Gamma\Gamma'\right)^{-1}M,$$

where the interest rate r is already contained into the matrix M (see Section 2). Accordingly, in our framework, we are not able to disentangle the two risks linked to the interest rate r and to the market price of risk. Furthermore, if we try to distinguish the terms in r, then we find a second degree polynomial in r. This is due to the insertion of the inflation risk. In fact, here, the riskless asset becomes like the other risky assets, and the risk linked to the interest rate r becomes a component of the market price of risk. Nevertheless, the qualitative result after Lioui and Poncet is preserved.

---

[4]We outline that they define an investor who is endowed with a portfolio of discount bonds that he chooses not to trade until his investment horizon (H). This hypothesis allows the authors to have a non-zero first portfolio component $w_{(1)}^*$.

4 Conclusion

In this paper we have analysed the optimal portfolio problem for a defined contribution pension fund maximizing the expected value of its terminal utility function. The shareholders contribute a constant percentage of their salaries into the fund. The fund manager faces two kind of risks: the risk linked to the shareholders' salaries, which are supposed to be stochastic, and the risk linked to the inflation stochastic process. On the financial market there are a stock, a bond, and a riskless asset.

Without specifying any functional form for the coefficients of the stochastic processes considered in our model, we are able to find a close form solution to the asset allocation problem thanks to the introduction of the inflation risk. The inflation process changes the riskless asset into a risky asset, whose drift is given by the riskless interest rate net to inflation, and whose diffusion term is given by the inflation diffusion. Accordingly, the new market structure contains three risky assets.

The optimal portfolio is formed by three components. The first one is proportional to both the portfolio Sharpe ratio and the inverse of Arrow-Pratt risk aversion index. The second component depends on the interest rate parameters. The third component is preference free and depends only on the diffusion terms of assets and background risks.

We find that, in the second optimal portfolio component, the stock does not play any role, while the weights of the riskless asset and the bond are identical with opposite sign.

A   The Cholesky Decomposition of the Correlation Matrix

Let $\begin{bmatrix} W_x(t) & W_y(t) \end{bmatrix}'$ denote a vector of two independent standard Wiener processes. Thus, we have

$$cov\left[dW_x dW_y\right] = \mathbb{E}\left[dW_x dW_y\right] = 0,$$

and variance-covariance matrix $\Sigma = tI_{(2)}$, where $I_{(2)}$ denotes the identity matrix of dimension two. We can transform $\begin{bmatrix} W_x & W_y \end{bmatrix}'$ into a vector of two correlated Wiener processes $\begin{bmatrix} \widetilde{W}_x & \widetilde{W}_y \end{bmatrix}'$ with the same mean (i.e. zero mean) but with variance-covariance matrix

$$\widetilde{\Sigma} = \begin{bmatrix} \sigma_x^2 & \varphi\sigma_x\sigma_y \\ \varphi\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix},$$

by applying to the original vector of uncorrelated processes the Cholesky decomposition as follows

$$\begin{bmatrix} \widetilde{W}_x \\ \widetilde{W}_y \end{bmatrix} = C'_{\widetilde{\Sigma}} \begin{bmatrix} W_x \\ W_y \end{bmatrix},$$

where $C_{\widetilde{\Sigma}}$ is just the Cholesky decomposition of the matrix $\widetilde{\Sigma}$. The matrix $C_{\widetilde{\Sigma}}$ is an upper-triangular matrix such that $\widetilde{\Sigma} = C'_{\widetilde{\Sigma}} C_{\widetilde{\Sigma}}$. Finally, we have

$$\begin{bmatrix} \widetilde{W}_x \\ \widetilde{W}_y \end{bmatrix} = \begin{bmatrix} \sigma_x & \varphi\sigma_y \\ 0 & \sigma_y\sqrt{1-\varphi^2} \end{bmatrix}' \begin{bmatrix} W_x \\ W_y \end{bmatrix} = \begin{bmatrix} \sigma_x W_x \\ \sigma_y\varphi W_x + \sigma_y\sqrt{1-\varphi^2}W_y \end{bmatrix}.$$

In conclusion, the following general result holds: given a set of Wiener processes, correlated or uncorrelated, it can always be represented as a vector of Wiener processes with the same drift of the initial processes, and diffusion term equal to the transpose of the Cholesky matrix calculated with respect to the variance-covariance matrix of the initial processes.

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