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**MODELLING THE SURRENDER CONDITIONS IN
EQUITY-LINKED LIFE INSURANCE**

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ABSTRACT

We propose a model for pricing a unit-linked life insurance policy embedding a surrender option. We consider both single and annual premium contracts. First we analyse a quite general contract, for which we obtain a backward recursive valuation formula based on the Cox, Ross and Rubinstein (1979) *binomial* model. Then we concentrate upon a particular case, that is the famous model with *exogenous* minimum guarantees. In this case we extend our previous analysis in order to take into account the possibility that the guarantees at death or maturity and the surrender values are *endogenously* determined, and provide necessary and sufficient conditions for the premiums to be *well defined*.

Keywords: surrender option, equity-linked life insurance, exogenous and endogenous guarantees, single and annual premium contracts, binomial trees.

JEL Classification: C61, G13, G22 and G23.

Subject Categories: IE10, IE50, IB10.

1 Introduction

The surrender option embedded in several types of life insurance contracts gives the policyholder the right to early terminate the contract and to receive a cash amount, called *surrender value*. To avoid adverse selection phenomena, this option is usually granted only if the contract provides benefits both in case of death and in case of survival such as, e.g., endowment and whole-life insurance policies.

The problem of fixing the surrender conditions when designing a new policy is very important, specially if the financial component of the policy is predominant. An over-simplified way to solve the problem could be that of fixing very low or even

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null surrender values. The (only) advantage of this solution is that the insurance company can completely ignore the surrender option and use consolidated actuarial techniques for pricing (and hedging) the contract. After all, the surrender is a unilateral decision of the policyholder that does not respect the initial terms of the contract and implies a loss of future gains expected by the insurance company. However, this solution may have a disastrous effect from a marketability point of view. First of all, it may spread discontent through those that originally did not consider the possibility of surrender but are forced to abandon their contracts for unexpected events and now feel swindled. Secondly, markets are populated also by investors that do not know exactly their time horizon, for which the surrender conditions may constitute one of the key-elements in the choice of an investment product such as a life insurance contract. For these and similar reasons the insurance company could decide to fix very competitive surrender conditions, but in this case it cannot afford to ignore the surrender option, that constitutes a component of the contract and must be suitably rewarded. Then an accurate valuation of this option, along with the other elements of the contract, is called for.

In particular, given the surrender conditions, the valuation of such option can be performed by following two different approaches:

i) According to the former, the surrender decision is treated just like death, i.e., it is considered an “exogenous” cause of termination of the contract. Actually, such decision can be driven by several “personal” reasons out of the control of the insurance company. For instance, the policyholder can fall into financial difficulties, or he/she can become acquainted with a change in the health status of the insured, checked by the insurance company only at inception. The collection of sufficient statistics on surrenders (called also *withdrawals*) allows to estimate the expected surrender rates and to construct a multidecrement table with two possible causes of elimination: surrender and death. A very natural assumption combined with this approach is the stochastic independence between surrenders and the financial elements.

ii) The latter approach merges the contract into a *contingent-claims* framework, characterized by perfectly competitive and frictionless markets, populated by rational and non-satiated agents all sharing the same information. According to this approach, the surrender decision is not at all independent of the financial elements, since it is the consequence of a rational choice. Then, in this framework, the whole contract and, in particular, the surrender option, are typical *American-style*¹ contingent-claims, whose valuation is obtained by merging together the traditional actuarial techniques, based on *pooling* arguments, with the modern financial toolbox, based on the *no-arbitrage* principle.

The above two approaches may appear to be completely incompatible. Nevertheless we are fully convinced that, although most policyholders are very likely induced to surrender their policies by “exogenous” reasons, the correct approach to follow in the valuation of the contract is the second one, because the *fair* value of a right, such as the surrender option, is independent of the behaviour of its owner. In other words, the policyholder has the right to act “optimally” when taking his/her

¹We mean that the right to abandon the contract can be exercised at any time before its natural termination, that is death or maturity.

surrender decision, hence no “discounts” are admitted for anticipated non-optimal behaviours. In fact, even if such discounts were allowed, the insurance company could not subsequently forbid the policyholder to act optimally, and this could obviously turn out to be a serious threat to solvency.

The literature concerning the valuation of the surrender option in a contingent-claims framework is not very abundant. The only papers of which we are aware are those by Albizzati and Geman (1994), Grosen and Jørgensen (1997, 2000), Jensen, Jørgensen and Grosen (2001), Steffensen (2002), Bacinello (2003a, 2003b), Tanskanen and Lukkarinen (2003), and Vannucci (2003a, 2003b). Apart from the paper by Steffensen (2002), that acts in a very general framework, the above papers deal only with single premium contracts (except Bacinello (2003b)) and traditional or participating life insurance (except Grosen and Jørgensen (1997) and Vannucci (2003a, 2003b)), and analyse a *fixed-term* life insurance policy without mortality risk, hence a *purely-financial* contract (except Bacinello (2003a, 2003b) and Vannucci (2003a, 2003b)). The introduction of mortality risk is a very delicate point, even under the usual assumption of stochastic independence between mortality and financial factors. In fact, unlike European-style² contracts, in the valuation of American contracts it is not possible to keep separate the financial elements from mortality, because the surrender decision involves continuous comparisons between the surrender value and the value of the contract, that obviously depends also on mortality factors. Hence there is a continuous interaction between mortality and financial factors. Another interaction arises when the analysis is shifted from single-premium to periodic-premium contracts, because the periodic premium depends on the value of the surrender option, that in turn depends on the premium amount, even if the guarantees and the surrender values are exogenously given.

In this paper we propose a model for pricing, according to the second approach, a unit-linked life insurance contract embedding a surrender option. The contract analysed is an *endowment* policy. We have chosen such type of policy because it is characterized by a high level of savings component and delivers benefits both in case of death and in case of survival at maturity. Moreover, our analysis can be straightforwardly applied also to *whole-life* insurance by simply taking, as maturity date, the date corresponding to the terminal age of the insured³. We consider both single-premium and annual-premium payments. In a first moment we analyse a quite general equity-linked contract, without specifying the way in which benefits and surrender values are linked to the reference portfolio. For this general contract we obtain a backward recursive valuation formula based on the Cox, Ross and Rubinstein (1979) *binomial* model. Then we concentrate upon a particular case of the general contract, that is the famous model with *exogenous* minimum guarantees pioneered by Brennan and Schwartz (1976) and Boyle and Schwartz (1977). In this particular case we extend our previous analysis in order to take into account the possibility that the minimum guarantees at death or maturity and the cash surrender

²We mean contracts without the surrender option.

³In actuarial practice the survival probabilities are extracted from a life table, hence there is an age, usually referred to as *terminal age* and denoted by ω , such that the survival probabilities until any age $\geq \omega$ are = 0 independently of the current age of the insured (of course, provided that it is $< \omega$).

values are *endogenously* determined, and provide necessary and sufficient conditions for the premiums to be *well defined*. Here the terms *exogenous* and *endogenous* are used with the same meaning given them by Bacinello and Ortu (1993).

The paper is structured as follows. In Section 2 we introduce our valuation framework and analyse the single premium contract, first of all in general terms and after by endogenizing the minimum guarantees and the surrender values. The same analysis is then extended, in Section 3, to periodic premium contracts. The whole discussion is also accompanied by the presentation of some numerical results.

2 Single premium contracts

We describe, first of all, the structure of the contract; after that we introduce the valuation set-up and present our valuation procedure in a very general case. Then we extend the model to the case of endogenous guarantees and endogenous surrender values, and finally we discuss some numerical results.

2.1 The structure of the contract

Consider an equity-linked endowment policy issued at time 0 and maturing at time T . Assume that the time is measured in years, and that T is an integer. We denote by x the age of the insured at inception. Assume that the policy is paid by a single premium at issuance and that a fixed amount is immediately invested (or *deemed to be invested*) in a reference fund. As is very common in practice, we assume this is a traded mutual fund, split into units, that does not pay any dividend. We denote by U the single premium (to be determined), D the amount deemed to be invested in the reference fund and S_t the unit price of this fund at time t (≥ 0), with $S_0 > 0$. Hence the number of units of the fund acquired at time 0 is given by

$$n = \frac{D}{S_0} \tag{1}$$

and remains constant over time. The value of the (accumulated) investments in the reference fund at time t is then given by

$$F_t = nS_t = D \frac{S_t}{S_0}, \quad 0 < t \leq T \tag{2}$$

and depends only on the current unit price of the fund (besides the initial price S_0).

We recall that, in an endowment policy, the insurance company commits oneself to pay a benefit at death of the insured or at maturity, whichever comes first. We denote by C_t^M the benefit paid at time $t \leq T$ in case of death, and by C_T^V the benefit paid at maturity if the insured is still alive. Obviously, these liabilities are due only if the contract is still in force, i.e., if it has not been previously surrendered. In case of surrender at time t , $0 < t < T$, we assume that the company pays an amount denoted by R_t . Of course the surrender option can be exercised only if the insured is still alive.

The equity-linked feature implies that the benefit, at death or maturity, is linked to the current value of the accumulated investments in the reference fund, i.e., that

$$C_t^M = f_t^M(F_t), \quad 0 < t \leq T \quad \text{and} \quad C_T^V = f_T^V(F_T), \quad (3)$$

where f_t^M , $0 < t \leq T$, and f_T^V are suitable functions. We admit also the possibility that either f_t^M or f_T^V (but not both) are constant functions, i.e., that the benefit at death is fixed and only the benefit at maturity is linked to the reference fund, or vice versa.

EXAMPLES

The following examples could apply either to the benefit at death or to the benefit at maturity or to both. For this reason we do not specify the superscript of the function involved.

a) $f_t(F_t) = F_t$.

The benefit is given by the current value of the accumulated investments in the reference fund.

b) $f_t(F_t) = \max \{F_t, G_t\}$.

Here the benefit is given by the current value of the reference portfolio, provided that it does not fall below a minimum amount guaranteed, denoted by G_t . This is the classical example dealt with by Brennan and Schwartz (1976) and Boyle and Schwartz (1977), who were the first to recognize that such benefit can be expressed in terms of the final payoff of European call or put options on the value of the reference portfolio with strike the minimum amount guaranteed:

$$f_t(F_t) = G_t + \max \{F_t - G_t, 0\} = F_t + \max \{G_t - F_t, 0\}.$$

c) $f_t(F_t) = \min \{F_t, M_t\}$.

Here the benefit is capped by the amount M_t . Also in this case it can be decomposed in terms of call or put options:

$$f_t(F_t) = F_t - \max \{F_t - M_t, 0\} = M_t - \max \{M_t - F_t, 0\}.$$

d) $f_t(F_t) = \max \{\min \{F_t, M_t\}, G_t\} = \min \{\max \{F_t, G_t\}, M_t\}$.

Here the benefit is capped by M_t and there is also a minimum guarantee, given by $G_t (< M_t)$. In this case there are three alternative decompositions in terms of European options:

$$\begin{aligned} f_t(F_t) &= F_t + \max \{G_t - F_t, 0\} - \max \{F_t - M_t, 0\} \\ &= G_t + \max \{F_t - G_t, 0\} - \max \{F_t - M_t, 0\} \\ &= M_t + \max \{G_t - F_t, 0\} - \max \{M_t - F_t, 0\}. \end{aligned}$$

Whichever the particular case considered may be, we emphasize that all fixed amounts intervening in the definition of the functions f_t^M and f_T^V (such as, e.g., G_t and M_t) are assumed, at least for the moment, to be independent of the premium U , i.e., exogenously given.

The surrender values R_t could either be fixed (exogenously) or specified in a similar way. To remain quite general, we assume then

$$R_t = g_t(F_t), \quad 0 < t < T. \quad (4)$$

2.2 The valuation set-up

We assume to act in perfectly competitive and frictionless markets, free of arbitrage opportunities, where all the agents are rational and non-satiated and share the same (financial) information.

We assume that the rate of return on risk-free assets is deterministic and constant and denote it by r . This is undoubtedly a great limit of our model. Considering that life insurance policies are usually long-term contracts, the introduction of stochastic interest rates would be desirable. However, in this paper, we have chosen to concentrate only upon the surrender feature in order to keep the model simpler. As we will see, in spite of this ease the computational complexity of our valuation formulae remains considerable, specially when periodic premium contracts are dealt with.

We assume stochastic independence between the lifetime of the insured and the unit value of the reference fund, and that the insurance company is risk-neutral with respect to mortality. This assumption means that it does not request any compensation for assuming mortality risk, and is usually justified by pooling arguments. As a matter of fact, insurance companies are not risk-neutral with respect to mortality due to the longevity risk impending over their whole portfolios, i.e., to the risk of systematic (negative) deviations between actual and expected mortality rates. However, since we are dealing with an endowment policy, if the benefit at death and the benefit in case of survival at maturity are comparable, the longevity risk should not be a problem because it simply tends to postpone the payment date.

As far as the financial uncertainty is concerned, we adopt the discrete model by Cox, Ross and Rubinstein (1979). More in detail, we divide the time interval $[0, T]$ into K sub-intervals of equal length $\Delta = T/K$, and assume that financial markets are open only at the discrete dates $t = k\Delta$, $k = 0, 1, \dots$. Then, given two real numbers u and d such that $0 < d < e^{r\Delta} < u$ and a generic instant $t = k\Delta$, we assume that, conditional to the information available at time t , the unit price of the reference fund at time $t + \Delta$ can take only two possible values, an “up” value, given by uS_t , and a “down” value, given by dS_t :

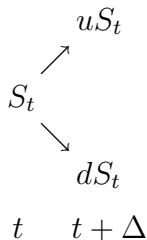


FIGURE 1: *The binomial model*

As is well known, in this discrete setting absence of arbitrage is equivalent to the existence of a risk-neutral measure that turns into martingales the price processes after discounting with the risk-free rate r . We recall that, under this measure, the events $\left\{\frac{S_{t+\Delta}}{S_t} = u\right\}$ and $\left\{\frac{S_{t+\Delta}}{S_t} = d\right\}$, independent of S_t , have (strictly positive) probabilities

$$q = \frac{e^{r\Delta} - d}{u - d} \quad \text{and, respectively,} \quad 1 - q = \frac{u - e^{r\Delta}}{u - d}. \quad (5)$$

The above discretization requires to better specify some previous assumptions. More precisely, we assume that

- in case of death of the insured between times $k\Delta$ and $(k+1)\Delta$ the benefit $C_{(k+1)\Delta}^M$ is paid at time $(k+1)\Delta$, $k = 0, 1, \dots, K-1$;
- the surrender decision is considered at the beginning of each sub-interval, i.e., at times $k\Delta$, $k = 1, 2, \dots, K-1$.

2.3 The valuation procedure

The stochastic evolution of the unit price S_t during the life of the contract can be represented in a binomial tree with *recombining* nodes:

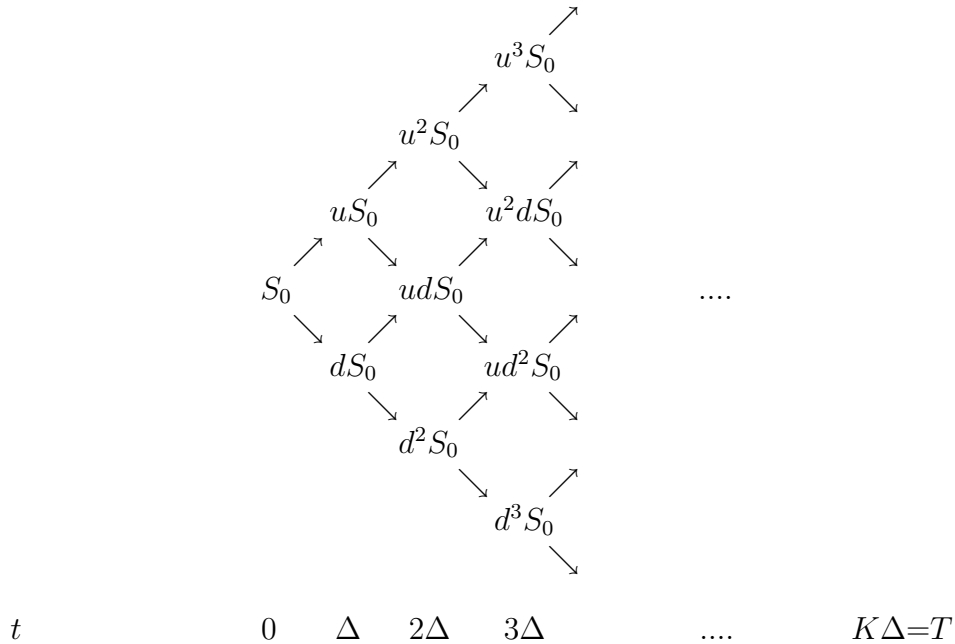


FIGURE 2: *The stochastic evolution of the unit price S_t*

From time 0 to time $k\Delta$, $k = 1, 2, \dots, K$, there are 2^k different paths for S_t , that however lead to only $k+1$ different values, given by

$$S_{k\Delta}^i = u^{k-i} d^i S_0, \quad k = 1, 2, \dots, K \quad \text{and} \quad i = 0, 1, \dots, k. \quad (6)$$

In fact, for any $k = 1, 2, \dots, K$ and $i = 0, 1, \dots, k$, there are $\binom{k}{i}$ different paths that lead to the same value $S_{k\Delta}^i$.

Recalling that F_t is proportional to S_t (see relation (2)), in a similar tree we can imagine to represent

- the value of the accumulated investments in the reference fund $F_{k\Delta} = nS_{k\Delta}$, $k = 1, 2, \dots, K$,
- the benefit at death $C_{k\Delta}^M = f_{k\Delta}^M(F_{k\Delta})$, $k = 1, 2, \dots, K$,
- the benefit at maturity $C_{K\Delta}^V = f_{K\Delta}^V(F_{K\Delta})$,
- the surrender value $R_{k\Delta} = g_{k\Delta}(F_{k\Delta})$, $k = 1, 2, \dots, K - 1$,

and, moreover:

- the current value of the “whole contract” (including the compensation for the surrender option), that we denote by $V_{k\Delta}$, $k = 0, 1, \dots, K - 1$,
- a “continuation” value, denoted by $W_{k\Delta}$, $k = 0, 1, \dots, K - 1$, that we are going to define immediately.

Of course, the “fair” single premium of the contract is given by

$$U = V_0. \quad (7)$$

To compute V and W we proceed by backward induction as follows:

1. Assume first of all that the insured is still alive at the beginning of the last sub-interval, i.e., at time $t = (K - 1)\Delta = T - \Delta$, that the contract has not been previously surrendered and that the current unit price of the reference fund is $S_{T-\Delta}^i$ ($i = 0, 1, \dots, K - 1$). Then the policyholder has two alternatives:
 - (a) to surrender the contract, and in this case he/she immediately receives the surrender value $R_{T-\Delta}^i \equiv g_{T-\Delta}(nS_{T-\Delta}^i)$;
 - (b) to continue the contract, and in this case the beneficiary will receive, at time T , the (*stochastic*) benefit C_T^M if the insured dies during the last time interval or, alternatively, the (*stochastic*) benefit C_T^V if the insured survives the maturity date.

Recalling that, conditional to the information available at time $T - \Delta$, C_T^M and C_T^V can take only two possible values, given by

$$C_T^h = \begin{cases} f_T^h(nuS_{T-\Delta}^i) & \text{if } S_T = uS_{T-\Delta}^i, \\ f_T^h(ndS_{T-\Delta}^i) & \text{if } S_T = dS_{T-\Delta}^i, \end{cases} \quad h = M, V,$$

⁴Note that this value represents also the reserve that the insurance company should set aside for the contract.

we have that the (contingent) continuation value is given by

$$\begin{aligned}
W_{T-\Delta} = W_{T-\Delta}^i &\equiv \Delta q_{x+T-\Delta} \{e^{-r\Delta} [q f_T^M(nu S_{T-\Delta}^i) + (1-q) f_T^M(nd S_{T-\Delta}^i)]\} \\
&+ \Delta p_{x+T-\Delta} \{e^{-r\Delta} [q f_T^V(nu S_{T-\Delta}^i) + (1-q) f_T^V(nd S_{T-\Delta}^i)]\}, \\
i &= 0, 1, \dots, K-1.
\end{aligned} \tag{8}$$

Here $\Delta q_{x+T-\Delta}$ and $\Delta p_{x+T-\Delta}$ represent the probabilities, assigned by the insurance company, that the insured dies in the last sub-period or survives the maturity date, both conditioned on survival at time $T-\Delta$.

Observe that $W_{T-\Delta}$ is computed as a joint risk-neutral conditional expectation, with respect to both financial and mortality uncertainty, of the discounted final “payoff” of the contract.

Finally, we define the value of the whole contract as

$$V_{T-\Delta} = V_{T-\Delta}^i \equiv \max \{R_{T-\Delta}^i, W_{T-\Delta}^i\}, \quad i = 0, 1, \dots, K-1, \tag{9}$$

since any rational and non-satiated policyholder is assumed to behave in order to maximize his(her) profit.

2. Assume now that the insured is still alive at time $t = k\Delta$, $k = 0, 1, \dots, K-2$ (hence $t < T-\Delta$ is the beginning of a generic sub-period except the last one), that the contract has not been previously surrendered and that the current unit price of the reference fund is $S_{k\Delta}^i$ ($i = 0, 1, \dots, k$). Once again the policyholder has two alternatives:

- (a) to surrender the contract, and in this case he(she) immediately receives the surrender value $R_{k\Delta}^i \equiv g_{k\Delta}(n S_{k\Delta}^i)$ ⁵;
- (b) to continue the contract, and in this case the beneficiary will receive, at time $(k+1)\Delta$, the (*stochastic*) benefit $C_{(k+1)\Delta}^M$ if the insured dies between times $k\Delta$ and $(k+1)\Delta$ or, alternatively, the policyholder will be entitled to a contract with (*stochastic*) value, at time $(k+1)\Delta$, given by $V_{(k+1)\Delta}$.

Recall that, conditional to the information available at time $k\Delta$, $C_{(k+1)\Delta}^M$ and $V_{(k+1)\Delta}$ can take only two possible values, respectively given by

$$C_{(k+1)\Delta}^M = \begin{cases} f_{(k+1)\Delta}^M(nu S_{k\Delta}^i) & \text{if } S_{(k+1)\Delta} = u S_{k\Delta}^i \\ f_{(k+1)\Delta}^M(nd S_{k\Delta}^i) & \text{if } S_{(k+1)\Delta} = d S_{k\Delta}^i \end{cases}$$

and

$$V_{(k+1)\Delta} = \begin{cases} V_{(k+1)\Delta}^i & \text{if } S_{(k+1)\Delta} = u S_{k\Delta}^i \\ V_{(k+1)\Delta}^{i+1} & \text{if } S_{(k+1)\Delta} = d S_{k\Delta}^i \end{cases},$$

⁵Actually we do not admit this first alternative when $k=0$, i.e., immediately after the payment of the single premium U . This reasonable assumption will become important, from a technical point of view, both in the case of annual premiums and in that of endogenous guarantees.

where the values $V_{(k+1)\Delta}^i$ and $V_{(k+1)\Delta}^{i+1}$ are those constructed in the preceding iterative step. Then the continuation value is given by

$$\begin{aligned} W_{k\Delta} = W_{k\Delta}^i &\equiv \Delta q_{x+k\Delta} \{e^{-r\Delta} [q f_{(k+1)\Delta}^M (nu S_{k\Delta}^i) + (1-q) f_{(k+1)\Delta}^M (nd S_{k\Delta}^i)]\} \\ &\quad + \Delta p_{x+k\Delta} \{e^{-r\Delta} [q V_{(k+1)\Delta}^i + (1-q) V_{(k+1)\Delta}^{i+1}]\}, \\ k &= 0, 1, \dots, K-2 \quad \text{and} \quad i = 0, 1, \dots, k, \end{aligned} \quad (10)$$

and the value of the whole contract by

$$V_{k\Delta} = V_{k\Delta}^i \equiv \begin{cases} W_0^0 & \text{if } k=0 \\ \max \{R_{k\Delta}^i, W_{k\Delta}^i\} & \text{if } k=1, 2, \dots, K-2 \text{ and } i=0, 1, \dots, k \end{cases}. \quad (11)$$

Once again, $\Delta q_{x+k\Delta}$ and $\Delta p_{x+k\Delta}$ represent the conditional probabilities of death or survival assigned by the insurance company on a collective basis.

Summing up, the time 0 value of the contract, and hence the single premium U , can be calculated by backward induction, applying first of all relations (8) and (9) for $i = 0, 1, \dots, K-1$, and after relations (10) and (11) for $k = K-2, K-1, \dots, 0$ and $i = 0, 1, \dots, k$. From a computational point of view, this valuation procedure is characterized by a quadratic complexity, since the total nodes of the binomial tree to be “visited” in order to compute U are $1 + 2 + \dots + K = \frac{K(K+1)}{2}$.

REMARK 1

Observe that relations (9) and (11) imply a threshold for the surrender value, given by $W_{k\Delta}$ ($k = 1, 2, \dots, K-1$), in the sense that the policyholder is assumed to surrender the contract as long as the surrender value $R_{k\Delta}$ exceeds the continuation value $W_{k\Delta}$. This threshold, defined by relations (8) and (10), is computed by using the risk-neutral financial probabilities (q and $1-q$) and the mortality and survival probabilities ($\Delta q_{x+k\Delta}$ and $\Delta p_{x+k\Delta}$) assigned by the insurance company, assumed to be risk-neutral with respect to mortality. While the financial probabilities are common knowledge of the insurance company and the policyholder because they derive from financial prices, the mortality probabilities assigned by the policyholder could be different from those assigned by the insurance company because the policyholder has an “insider” information of the health status of the insured⁶. In presence of mortality risk, i.e., when $C_T^M \neq C_T^V$ and $C_{k\Delta}^M \neq V_{k\Delta}$ ($k = 1, 2, \dots, K-1$) with strictly positive probability, this fact could obviously lead to a different continuation value for the policyholder, hence to a different threshold for surrender. Moreover, even if the insurance company and the policyholder agreed on the mortality probabilities, the actual threshold of the policyholder could be different from $W_{k\Delta}$ because the policyholder is not assumed to be risk-neutral with respect to mortality. In particular, in case of risk-aversion, he(she) could be willing to give up the surrender value in order to continue the contract even if it exceeds $W_{k\Delta}$, hence he(she) could

⁶Note that the policyholder and the insured are usually the same person.

present a higher threshold, and vice versa in case of risk-propensity. Nevertheless, we are fully convinced that relations (8) to (11) represent the correct way for pricing and reserving the contract. Assume, in fact, that the insurance company knows the “actual” threshold of the policyholder, given by $Z_{k\Delta}$. In this case the current value of its liabilities would be given by

$$Y_{k\Delta} = \begin{cases} W_{k\Delta} & \text{if } R_{k\Delta} \leq Z_{k\Delta} \\ R_{k\Delta} & \text{if } R_{k\Delta} > Z_{k\Delta} \end{cases} \leq \max \{W_{k\Delta}, R_{k\Delta}\} \quad \forall Z_{k\Delta} \neq W_{k\Delta}, \quad k = 1, 2, \dots, K-1.$$

Note that this expression exploits the assumed risk-neutrality with respect to mortality of the insurance company because the cost ascribed to continuation is still $W_{k\Delta}$, even if it is different from the continuation threshold $Z_{k\Delta}$. Since, in practice, the actual threshold $Z_{k\Delta}$ is neither known in advance nor can be imposed to the policyholder, solvency requirements justify the use of the previous relations (8) to (11).

REMARK 2

Our procedure supplies a way for computing the single premium of the whole contract, inclusive of the compensation for the surrender option. If the insurance company is interested to quantify, separately, the value of such option, in order to understand its incidence on the premium, it could first compute, along with the premium U , the time 0 value of the European version of the contract, say U^E , and after obtain the premium for the surrender option as the difference between U and U^E . To compute U^E it is possible to follow step by step our procedure, with the only difference that the value of the whole contract V_t defined in relations (9) and (11) must now be set equal to the continuation value W_t . It is easy to prove, by induction, that this modified procedure leads to the following expression for the premium U^E :

$$\begin{aligned} U^E &= \sum_{k=1}^K {}_{(k-1)\Delta|\Delta}q_x e^{-rk\Delta} \sum_{i=0}^k \binom{k}{i} q^{k-i} (1-q)^i f_{k\Delta}^M(nS_{k\Delta}^i) \\ &+ {}_T p_x e^{-rT} \sum_{i=0}^K \binom{K}{i} q^{K-i} (1-q)^i f_T^V(nS_T^i), \end{aligned} \quad (12)$$

where ${}_{(k-1)\Delta|\Delta}q_x$ denotes the (unconditional) probability that the insured dies between times $(k-1)\Delta$ and $k\Delta$, and ${}_T p_x$ is the probability that he/she survives the maturity date.

REMARK 3

- a) If $C_t^M = R_t = F_t$ for any t ($0 < t < T$) and $C_T^M = C_T^V = F_T$, then there is neither mortality nor investment risk for the insurance company if it actually invests at time 0 the amount D in the reference fund. In this case the surrender option is valueless, since $U = U^E = D$.

b) If $C_t^M = \max\{F_t, G_t\}$ for any t ($0 < t \leq T$) and $C_T^V = C_T^M$, then $W_t \geq F_t$ for any t ($0 < t < T$). Hence:

- i) if $R_t = F_t$ for any t ($0 < t < T$), then $V_t = \max\{R_t, W_t\} = W_t$ for any t , which implies a zero value for the surrender option;
- ii) if $R_t = \max\{F_t, H_t\}$ for any t ($0 < t < T$), where H_t is a fixed amount (no matter if coinciding or not with G_t), or, alternatively, $R_t = H_t$ for any t , then the surrender option has the same value. In fact, since $W_t \geq F_t$, $V_t = \max\{\max\{F_t, H_t\}, W_t\} = \max\{F_t, H_t, W_t\} = \max\{H_t, W_t\}$ for any t .

Note that, even if $H_t < G_t$ for any t , the surrender option could have a strictly positive value because H_t is never compared with G_t but, through the continuation value, it is compared with $G_{t+\tau}e^{-r\tau}$ for $0 < t < T$ and $0 < \tau \leq T - t$. A sufficient condition for the surrender option to be valueless is then $H_t \leq G_{t+\tau}e^{-r\tau}$ for any t, τ . If, for instance, $G_t = De^{gt}$ and $H_t = De^{ht}$, this condition is satisfied when $h < r$ and $g \geq \frac{h+r(K-1)}{K}$ or, alternatively, $h \geq r$ and $g \geq \frac{h(K-1)+r}{K}$.

2.4 Endogenous guarantees and surrender values

Assume now that the benefit at death or maturity is guaranteed, as in Remark 3b), as well as the surrender value. We have so far supposed that all the guarantees are fixed amounts, i.e., they are *exogenously* given. However, it is standard actuarial practice to offer guarantees and surrender values functionally dependent on the premium, hence *endogenously* determined. The rationale of this practice is that the policyholder usually considers the whole premium U as his(her) initial investment in the contract and does not realize that actually a part of U is not investment but charge for guarantees. For instance, if the guarantees at death, maturity or surrender are given by the amount D and the premium U turns out to be 125% of D , the policyholder simply perceives that the contract guarantees only 80% of his(her) initial investment, and this could be not good from a marketability point of view. Then, in what follows, we assume that the guarantees are endogenously specified. We do not consider quite general guarantee functions, but restrict ourselves to the most significant practical example, that is

$$C_t^M = R_t = \max\{F_t, Ue^{\delta t}\}, \quad \delta \geq 0 \quad \text{and} \quad 0 < t < T \quad (13)$$

and

$$C_T^M = C_T^V = \max\{F_T, Ue^{\delta T}\}, \quad \delta \geq 0, \quad (14)$$

where δ is a minimum interest rate guaranteed. As observed in Remark 3b)ii), our analysis can be straightly applied also to the case in which the surrender values are simply given by $Ue^{\delta t}$.

The specification of endogenous guarantees makes the time 0 value of the whole contract, V_0 , defined in our recursive valuation procedure (8)-(11), be a function of U , say $f(U)$. Hence our goal is to give conditions under which the premium is well defined, i.e., there exists a unique (strictly positive) U such that

$$f(U) = U \quad \text{or, equivalently,} \quad g(U) \equiv f(U) - U = 0. \quad (15)$$

Our problem can then be translated into the existence (and uniqueness) of a fixed point of the function f or, equivalently, a zero of the function g . The following proposition provides necessary and sufficient conditions for this.

Proposition 1

A necessary and sufficient condition for the existence and uniqueness of a premium U satisfying relation (15) is $\delta < r$.

Proof

Necessity:

Assume that there exists a premium U satisfying (15). Ab absurd, assume moreover that $\delta > r$. Then the policyholder could borrow U at the risk-free rate r and buy the contract. At surrender, or death, or maturity, he/she (or the beneficiary) would receive at least the guaranteed amount U plus accrued interests at rate δ , with which he/she could pay off his/her debt and realize (at least) a strictly positive interest gain. This is obviously impossible, since arbitrage opportunities are ruled out of the market, hence δ must be $\leq r$. However, if $\delta = r$, it is easy to verify that $f(U) = U$ for any $U \geq Du^K e^{-rT}$. In fact, in these cases, the guarantees become always effective, whatever the time of death or surrender may be, because $Ue^{rk\Delta} \geq nS_{k\Delta}^i$ for any $k = 1, 2, \dots, K$ and $i = 0, 1, \dots, k$. Hence the uniqueness of the solution implies $\delta \neq r$.

Sufficiency:

Observe, first of all, that the function f is continuous on all non-negative real numbers, and that

$$g(0) = f(0) = D > 0$$

because this corresponds to the case of no guarantees (see Remark 3a). Moreover, for any $U \geq Du^K e^{-\delta T}$, it is easy to prove that our recursive procedure leads to $f(U) = Ue^{(\delta-r)\Delta}$, since also in these cases the guarantees become always effective, whatever the time of death or surrender may be. This last fact, along with the condition $\delta < r$, implies

$$\lim_{U \rightarrow +\infty} g(U) = \lim_{U \rightarrow +\infty} U [e^{(\delta-r)\Delta} - 1] = -\infty.$$

Hence the existence of a zero for g is guaranteed by the properties of continuous functions.

The proof of uniqueness can be supplied by exploiting either a property of convex functions or, alternatively, pure-arbitrage arguments.

Following the first approach, it is immediate to verify that the function f is (weakly) convex. Then, if it admitted two different fixed points, say U_1 and U_2 with $U_1 < U_2$, the incremental ratio $\frac{f(U)-f(U_1)}{U-U_1} = \frac{f(U)-U_1}{U-U_1}$ would be $=1$ for $U=U_2$ and, due to convexity, ≥ 1 for any $U \geq U_2$. This would imply $f(U) \geq U$ for any $U \geq U_2$, hence $\lim_{U \rightarrow +\infty} (f(U) - U)$ could not be $-\infty$, as previously established.

The second approach could be alternatively followed in order to prove that g is strictly decreasing (which obviously implies uniqueness of its zero). To see this

assume, ab absurd, that there exist two non-negative values U_1 and U_2 such that $U_1 < U_2$ and $f(U_1) - U_1 \leq f(U_2) - U_2$. Then the insurance company could

- a) sell a contract with exogenous guaranteed benefits and surrender values given by U_2 plus accrued interests at rate δ , receiving from the policyholder the single premium $f(U_2)$;
- b) buy from a reinsurer an identical contract on the same life, but with guaranteed benefits and surrender values given by U_1 plus interests, paying the single premium $f(U_1)$ ⁷;
- c) invest in the riskless asset the amount $U_2 - U_1$.

This strategy would produce a non-negative inflow at time 0. Assume that the insurance company surrenders its reinsurance contract only if and when the policyholder surrenders his(her) contract. Then, at time of termination, say $t (\leq T)$, the insurance company would

- a) pay the benefit (or the surrender value) $\max \{F_t, U_2 e^{\delta t}\}$ to the beneficiary (or to the policyholder) of the contract sold;
- b) receive $\max \{F_t, U_1 e^{\delta t}\}$ from the reinsurer;
- c) sell back the riskless asset, collecting the amount $(U_2 - U_1)e^{rt}$.

The final payoff of this strategy, given by

$$\begin{aligned} & \max \{F_t, U_1 e^{\delta t}\} - \max \{F_t, U_2 e^{\delta t}\} + (U_2 - U_1)e^{rt} \\ &= \begin{cases} (U_2 - U_1)(e^{rt} - e^{\delta t}) & \text{if } F_t \leq U_1 e^{\delta t} \\ F_t - U_2 e^{\delta t} + (U_2 - U_1)e^{rt} > (U_2 - U_1)(e^{rt} - e^{\delta t}) & \text{if } U_1 e^{\delta t} < F_t < U_2 e^{\delta t} , \\ (U_2 - U_1)e^{rt} & \text{if } F_t \geq U_2 e^{\delta t} \end{cases} \end{aligned}$$

would be strictly positive whatever F_t may be, hence an arbitrage opportunity would exist \square

To conclude, we observe that the unique single premium U can be numerically computed by applying simple iterative methods.

2.5 Numerical results

In order to get a numerical feeling about the incidence of the premiums for the surrender option and for (possible) minimum guarantee provisions at death or maturity, in what follows we present the results of some numerical experiments. In all these experiments we have fixed $D=100$, so that the premiums for the guarantees at death or maturity and for the surrender option are expressed as percentages of the initial amount deemed to be invested in the reference fund. More in detail, we have computed, first of all, the single premium U^E for the European version of the

⁷Note that a part of the risk is retained by the primary insurer because $U_1 < U_2$.

contract, defined in relation (12), and after the premium U for the whole contract obtained by applying our recursive valuation procedure (8)-(11). Then, denoting by G the premium for possible guarantees at death or maturity and by H that for the surrender option, we have set

$$G = U^E - D \quad \text{and} \quad H = U - U^E. \quad (16)$$

Moreover, to estimate the mortality probabilities we have used the life table of the Italian Statistics for Males Mortality in 1991, with values corresponding to non-integer ages computed by linear interpolation. As far as the discretization is concerned, we have divided each year of contract into 100 intervals of equal length $\Delta=0.01$, so that $K=100T$. Finally, we have used the following parametrization (see also Hull (2003)):

$$u = e^{\sigma\sqrt{\Delta}} \quad \text{and} \quad d = \frac{1}{u}, \quad (17)$$

where σ characterizes the volatility of the unit price of the reference fund. To satisfy the condition $d < e^{r\Delta} < u$, we have always fixed $\sigma > r\sqrt{\Delta}$, that is $\sigma > r/10$.

Starting with the case of exogenous guarantees we represent, in Table 1, the results obtained when the surrender values and the benefits are fixed as in Remark 3b)ii) of Section 2.3, i.e.,

$$C_t^M = \max \{F_t, De^{gt}\}, \quad 0 < t \leq T \quad \text{and} \quad C_T^V = C_T^M,$$

$$R_t = \max \{F_t, De^{ht}\} \quad \text{or, equivalently,} \quad R_t = De^{ht}, \quad 0 < t < T$$

for different levels of the annual rates guaranteed g and h .

TABLE 1

The premium for the minimum guarantees (G) and for the surrender option (H) for different levels of the minimum interest rates guaranteed at death or maturity (g) and at surrender (h)

	$g = 0.00$	$g = 0.02$	$g = 0.04$	$g = 0.06$	$g = 0.08$	$g = 0.10$
	$G = 9.67$	$G = 19.13$	$G = 36.10$	$G = 65.20$	$G = 113.24$	$G = 189.94$
$h = 0.00$	$H = 12.23$	$H = 4.91$	$H = 0.03$	$H = 0.00$	$H = 0.00$	$H = 0.00$
$h = 0.02$	$H = 18.70$	$H = 9.35$	$H = 0.26$	$H = 0.00$	$H = 0.00$	$H = 0.00$
$h = 0.04$	$H = 29.36$	$H = 20.05$	$H = 3.41$	$H = 0.00$	$H = 0.00$	$H = 0.00$
$h = 0.06$	$H = 51.66$	$H = 42.97$	$H = 27.21$	$H = 0.00$	$H = 0.00$	$H = 0.00$
$h = 0.08$	$H = 96.79$	$H = 88.10$	$H = 72.34$	$H = 45.12$	$H = 0.00$	$H = 0.00$
$h = 0.10$	$H = 169.20$	$H = 160.51$	$H = 144.75$	$H = 117.53$	$H = 72.35$	$H = 0.00$

Note: $D=100$, $x=40$, $T=20$, $\Delta=0.01$, $r=0.05$, $\sigma=0.3$.

The results reported in Table 1 show that the prices for the minimum guarantee provisions and for the surrender option can be quite considerable, even when the guaranteed benefit at death or maturity and the surrender value are simply given by

the amount D without interests (that corresponds to the case $g=h=0$). In this case the total charge for guarantees (at death, maturity or surrender) is $G + H = 21.9\%$ of D , and this charge goes up to 39.51% when $g=h=4\%$. Note that, in the exogenous case, there are no constraints on g and h , that could also (very unrealistically) be greater than r , but similar provisions lead to extremely high values for the guarantees (189.94% of D when $g=10\%$!) and, moreover, the contract tends to lose its equity-linked feature. Finally, we observe that, when g is high (compared with r) and $h \leq g$, most (or even all) of the total charge is ascribed to minimum guarantees at death or maturity, while the (residual) surrender option has a very low (or even null) value. To isolate the effect of the surrender option, in the following Table 2 we report the results obtained when

$$C_t^M = F_t, \quad 0 < t \leq T \quad \text{and} \quad C_T^V = C_T^M,$$

and the surrender values are fixed as in the previous examples. In this case, in fact, $U^E = D$ (hence $H = U - D$) and all the charge is due to guarantees at surrender.

TABLE 2

The premium for the surrender option (H) in the case of no guarantees at death or maturity for different levels of the minimum rate guaranteed at surrender (h)

$h = 0.00$	$h = 0.02$	$h = 0.04$	$h = 0.06$	$h = 0.08$	$h = 0.10$
$H = 21.81$	$H = 28.25$	$H = 38.81$	$H = 60.20$	$H = 105.28$	$H = 177.62$

Note: $D=100$, $x=40$, $T=20$, $\Delta=0.01$, $r=0.05$, $\sigma=0.3$.

As expected, from the comparison between the results of Table 2 and those of Table 1 one can notice that now the surrender option absorbs almost all the total cost arising from guarantees at death, maturity and surrender.

Coming now to the endogenous case, i.e., to the case in which

$$C_t^M = R_t = \max \{F_t, Ue^{\delta t}\}, \quad 0 < t < T \quad \text{and} \quad C_T^V = C_T^M = \max \{F_T, Ue^{\delta T}\},$$

we have, first of all, fixed a basic set of parameters, given by $x=40$, $T=20$, $\delta=2\%$, $r=5\%$, $\sigma=25\%$, and then we have moved each parameter one at a time in order to catch some comparative statics properties of our model. The premiums obtained in the basic example, always expressed as percentages of D , are 20.63 for the minimum guarantees at death or maturity⁸, and 71.09 for the surrender option. Note that the premium for the surrender option is extremely high, even in this basic example. To understand the incidence of the endogenization we have also computed the premiums in the exogenous case, i.e., in the case in which the guarantees at death, maturity or surrender are given by D (instead of U) plus accrued interests at rate δ . In this case we have obtained 14.30 for the guarantees at death or maturity, and 8.28 for the surrender option, so that the total cost of endogenization is $E = (20.63 -$

⁸The existence of a unique premium in the endogenous case for the European version of the contract is guaranteed by the condition $\delta < r$ whatever the stochastic model followed by the unit price S_t may be (see Bacinello and Ortu (1993)).

$14.30) + (71.09 - 8.28) = 6.33 + 62.81 = 69.14$. Note that most of this cost is due to the endogenization of the guarantees at surrender. However, it must be pointed out that, while it is correct to consider the difference $20.63 - 14.30 = 6.33$ as the cost for endogenizing the guarantees at death or maturity in the European contract (i.e., without the surrender option), it is not completely correct to consider the difference $71.09 - 8.28 = 62.81$ as the additional cost needed for endogenizing the guarantees at surrender. In fact 8.28 is the value of the surrender option embedded in a contract with exogenous guarantees at death or maturity and exogenous surrender values, while 71.09 is the value of the surrender option embedded in a contract with endogenous guarantees at death or maturity and endogenous surrender values. The value of the surrender option embedded in a contract with endogenous guarantees at death or maturity and exogenous surrender values, always in this basic example, is instead equal to only 4.08, so that the cost for endogenizing the surrender values in the case of a contract with endogenous guarantees at death or maturity goes up to $71.09 - 4.08 = 67.01$ ⁹. To avoid confusion with too many figures, in the following Tables 3 to 7 we report only the premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, along with the total cost for endogenizing both the minimum guarantees at death or maturity and the guaranteed surrender values (E). The total premium for a contract with endogenous guarantees at death or maturity and endogenous surrender values is then

$$U = D + G + H + E.$$

TABLE 3

The premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (U) for different ages of the insured at inception (x)

x	G	H	E	U
30	14.26	8.44	69.45	192.15
31	14.27	8.44	69.43	192.14
32	14.27	8.43	69.42	192.12
33	14.27	8.42	69.41	192.10
34	14.28	8.41	69.38	192.07
35	14.28	8.40	69.36	192.04
36	14.28	8.38	69.33	191.99
37	14.29	8.36	69.29	191.94
38	14.29	8.34	69.25	191.88
39	14.30	8.31	69.19	191.80
40	14.30	8.28	69.14	191.72
41	14.31	8.25	69.06	191.62
42	14.31	8.21	68.99	191.51
43	14.32	8.17	68.90	191.39
44	14.32	8.12	68.82	191.26
45	14.33	8.07	68.71	191.11
46	14.33	8.02	68.59	190.94
47	14.34	7.95	68.47	190.76
48	14.35	7.89	68.33	190.57
49	14.36	7.81	68.19	190.36
50	14.36	7.74	68.02	190.12

Note: $D=100$, $T=20$, $\Delta=0.01$, $r=0.05$, $\delta=0.02$, $\sigma=0.25$.

⁹It is easy to extend the analysis of Section 3.4, in particular the sufficient condition of Proposition 1, to the case of endogenous guarantees at death or maturity and exogenous surrender values.

From Table 3 one can notice that the age of the insured has a very small effect on the premiums. All the quantities involved are decreasing, except G , that increases very slowly. The surrender option is not very expensive, but the cost of endogenization is always non-supportable.

TABLE 4

The premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (U) for different maturities (T)

T	G	H	E	U
10	15.52	4.12	62.07	181.71
11	15.55	4.56	63.21	183.32
12	15.51	5.01	64.21	184.73
13	15.44	5.44	65.10	185.98
14	15.33	5.87	65.89	187.09
15	15.20	6.30	66.59	188.09
16	15.05	6.71	67.21	188.97
17	14.88	7.12	67.77	189.77
18	14.70	7.52	68.27	190.49
19	14.51	7.90	68.73	191.14
20	14.30	8.28	69.14	191.72
21	14.10	8.64	69.51	192.25
22	13.89	9.00	69.84	192.73
23	13.68	9.34	70.14	193.16
24	13.47	9.67	70.41	193.55
25	13.26	9.99	70.66	193.91
26	13.05	10.29	70.89	194.23
27	12.85	10.59	71.07	194.51
28	12.65	10.87	71.26	194.78
29	12.46	11.13	71.42	195.01
30	12.27	11.39	71.56	195.22

Note: $D=100$, $x=40$, $\Delta=0.01$, $r=0.05$, $\delta=0.02$, $\sigma=0.25$.

From Table 4 one can notice that the maturity of the contract has a discrete influence on the premiums. In particular all the premiums are increasing (as expected), except G , that increases for T between 10 and 11 and after definitely decreases. The surrender option, H , is rather cheap for shorter maturities, specially if compared with G , but the cost of endogenization remains still non-sustainable.

The results reported in Table 5 show that the riskless rate has a very strong influence on the premiums, as expected. All the premiums are decreasing, except H , that increases until $r=8\%$ and after decreases. When r is very close to δ the premium G is rather high, while the (residual) surrender option has a very low value. In this case the cost of endogenization is absolutely out of control. When instead r is far from δ the sum $G+H$ is not very high, so that the endogenization, even if still very expensive, could be proposed to interested policyholders.

All the premiums reported in Table 6 are quickly increasing with respect to the minimum interest rate guaranteed δ , except the value of the surrender option, that instead decreases. Comparing the results of Tables 5 and 6 one can notice that all the premiums do not change with r and δ when the difference $r-\delta$ remains unchanged

(apart from very small differences due to approximation). This also happens to European contracts when the unit price of the reference fund follows the Black and Scholes (1973) and Merton (1973) model¹⁰. Recall that such model can be seen as the asymptotic version, for $\Delta \rightarrow 0$, of the discrete Cox, Ross and Rubinstein (1979) model adopted in this paper with parametrization defined as in relation (17). Then the numerical results show that this property is maintained also when the surrender option is inserted in the contract.

TABLE 5

The premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (U) for different levels of the riskless rate (r)

r	G	H	E	U
0.025	35.13	1.43	282.78	419.34
0.030	29.60	3.10	175.18	307.88
0.035	24.84	4.69	127.59	257.12
0.040	20.75	6.09	100.07	226.91
0.045	17.27	7.29	81.96	206.52
0.050	14.30	8.28	69.14	191.72
0.055	11.79	9.07	59.59	180.45
0.060	9.68	9.67	52.21	171.56
0.065	7.91	10.11	46.34	164.36
0.070	6.44	10.40	41.57	158.41
0.075	5.21	10.57	37.64	153.42
0.080	4.21	10.64	34.32	149.17
0.085	3.38	10.62	31.51	145.51
0.090	2.70	10.53	29.10	142.33
0.095	2.15	10.39	27.00	139.54
0.100	1.71	10.21	25.15	137.07

Note: $D=100$, $x=40$, $T=20$, $\Delta=0.01$, $\delta=0.02$, $\sigma=0.25$.

TABLE 6

The premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (U) for different levels of the minimum rate guaranteed (δ)

δ	G	H	E	U
0.000	6.44	10.40	41.51	158.35
0.005	7.91	10.11	46.31	164.33
0.010	9.68	9.67	52.19	171.54
0.015	11.79	9.07	59.58	180.44
0.020	14.30	8.28	69.14	191.72
0.025	17.26	7.29	81.98	206.53
0.030	20.75	6.10	100.07	226.92
0.035	24.83	4.69	127.60	257.12
0.040	29.59	3.11	175.17	307.87
0.045	35.12	1.43	282.73	419.28

Note: $D=100$, $x=40$, $T=20$, $\Delta=0.01$, $r=0.05$, $\sigma=0.25$.

¹⁰See Brennan and Schwartz (1976), Boyle and Schwartz (1977) and Bacinello and Ortu (1993).

TABLE 7

The premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (U) for different levels of the volatility parameter (σ)

σ	G	H	E	U
0.05	0.03	1.46	2.41	103.90
0.10	1.42	4.06	10.29	115.77
0.15	4.95	5.84	23.77	134.56
0.20	9.47	7.15	43.21	159.83
0.25	14.30	8.28	69.14	191.72
0.30	19.13	9.35	102.18	230.66
0.35	23.76	10.41	142.96	277.13
0.40	28.10	11.49	192.00	331.59
0.45	32.11	12.58	249.76	394.45
0.50	35.73	13.70	316.53	465.96

Note: $D=100$, $x=40$, $T=20$, $\Delta=0.01$, $r=0.05$, $\delta=0.02$.

Table 7 shows that, as expected, all the premiums are steeply increasing with the volatility parameter σ . When σ is low, it is absolutely proposable also the endogenization of guarantees and surrender values, but for medium-high levels of σ this becomes out of range.

Summing up, all the numerical results show that the endogenization of guarantees at death, maturity and surrender usually requires a very high cost, and remains feasible only if the volatility of the reference fund is low and the difference $r - \delta$ is high, i.e., if the contract is rather conservative. A way to keep the premium reasonable even for contracts not too much conservative could be that of endogenizing only the guarantees at death or maturity and letting exogenous the surrender values. This could sound as a sort of prize for faithful policyholders and should not compromise the marketability of the policy.

3 Periodic premium contracts

In this section we adapt our previous analysis to the case, more common in practice, in which the contract is paid by a sequence of constant premiums, due at the beginning of each year of contract, if the insured is still alive and the contract is still in force. Once again, we assume that a fixed amount, denoted by D , is deemed to be invested in a reference fund whenever an annual premium is paid. Our goal is then to determine a “fair” annual premium for the whole contract, inclusive of a compensation for the surrender option. Most of what said in the previous section applies straightforwardly to the case of periodic premium contracts, in particular the notation and examples of Section 2.1 and the valuation set-up described in Section 2.2. Hence, in what follows, we first highlight the main differences. Then we pass to redefine our valuation procedure and after to extend it to the case of endogenous guarantees. Finally, we discuss some numerical results.

3.1 Differences between single and annual premium contracts

A first, very important, difference arising when periodic premium contracts are dealt with is that the reference fund is “built” year by year. In fact the number of units acquired at time j is given by

$$n_j = \frac{D}{S_j}, \quad j = 0, 1, \dots, T - 1 \quad (18)$$

and the total number of units accumulated at time t is

$$N_t = \sum_{j=0}^{n(t)} n_j, \quad 0 < t \leq T, \quad (19)$$

where

$$n(t) = \begin{cases} t - 1 & \text{if } t \text{ is an integer} \\ \lfloor t \rfloor & \text{otherwise} \end{cases}, \quad (20)$$

with $\lfloor y \rfloor$ denoting the integer part of a real number y . Hence the value at time t of the accumulated investments in the reference fund is given by

$$F_t = N_t S_t = D \sum_{j=0}^{n(t)} \frac{S_t}{S_j}, \quad 0 < t \leq T \quad (21)$$

and depends not only on the current unit price but also on the unit prices at all premium payment dates¹¹. This *path-dependence* will increase remarkably the computational complexity of our valuation procedure. In fact all the variables involved cannot anymore be represented in a binomial tree with recombining nodes. To see this with a very simple example, assume that $K=T=3$, $u=2$, $d=1$, $S_0=1$ and $D=1000$. In this case, the stochastic evolution of F_t from times 1 to 3 can be represented as in Figure 3. All the 2^t ($t = 1, 2, 3$) trajectories that the unit price can follow from time 0 to time 3 need now to be considered, even if the different (final) values for it are only $t + 1$.

A second important difference is that both the continuation value and the value of the whole contract will be defined as “net” values, i.e., they will “measure” the future liabilities of the insurance company net of the future premiums to be collected from the policyholder¹². Unfortunately, this fact introduces a dependence between the above values and the premium, hence we denote them by $V_t(P)$ and $W_t(P)$, $0 \leq t < T$, where P is the annual premium. As a consequence, the time 0 value of the whole contract turns out to be a function of P , even in the case of exogenously given guarantees and surrender values.

¹¹Note that, if t is an integer, F_t is assumed to be valued immediately before the annual investment.

¹²Then the value of the whole contract will also in this case represent the reserve that the insurance company should set aside for the contract.

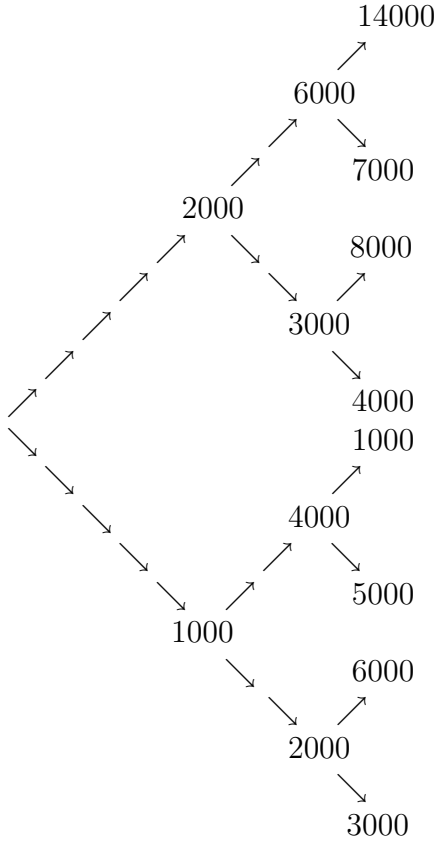


FIGURE 3: An example of stochastic evolution of F_t

3.2 The valuation procedure

Assume, first of all, that K , the number of intervals in which $[0, T]$ is divided, is a multiple of T , so that the premium payment dates correspond to times arising from our discretization. Moreover, assume that, when the beginning of a generic sub-interval coincides with a premium payment date, the surrender decision is considered before payment. This quite natural assumption is in accordance with the definition of F_t given in relation (21). Finally, imagine to represent the stochastic evolution of $F_{k\Delta}$, $C_{k\Delta}^M$ ($k = 1, 2, \dots, K$), $R_{k\Delta}$ ($k = 1, 2, \dots, K - 1$), $C_{K\Delta}^V$ and, for any given level of the annual premium P , $V_{k\Delta}(P)$, $W_{k\Delta}(P)$ ($k = 0, 1, \dots, K - 1$) in a binomial tree with *non-recombining* nodes (as in the example of Figure 3). To this end, assume to enumerate the nodes at time $k\Delta$ from 1 to 2^k , $k = 0, 1, \dots, K$, with the convention that, if $S_{k\Delta}^i$ is the unit value of the reference fund at time $k\Delta$, its two “following” values at time $(k + 1)\Delta$ are given by

$$S_{(k+1)\Delta}^{2i-1} = uS_{k\Delta}^i \quad \text{and} \quad S_{(k+1)\Delta}^{2i} = dS_{k\Delta}^i, \quad k = 0, 1, \dots, K - 1 \quad \text{and} \quad i = 1, 2, \dots, 2^k. \quad (22)$$

We stress that this notation is different from that used in the previous section, hence relation (6) does not apply anymore. Coherently with this notation, when necessary we will superscript the possible values of the variables involved.

Our valuation procedure requires, first of all, to “build” all the possible values of the accumulated investments in the reference fund, given by the following forward induction relation:

$$F_{(k+1)\Delta}^i = \begin{cases} u \left(F_{k\Delta}^{\frac{i+1}{2}} + D\mathbb{I}_{\{k\Delta \in \mathbb{N}\}} \right) & \text{if } i \text{ is odd} \\ d \left(F_{k\Delta}^{\frac{i}{2}} + D\mathbb{I}_{\{k\Delta \in \mathbb{N}\}} \right) & \text{if } i \text{ is even} \end{cases},$$

$$k = 0, 1, \dots, K - 1 \text{ and } i = 1, 2, \dots, 2^{k+1}, \quad (23)$$

where $F_0^1 \equiv 0$, and the indicator function $\mathbb{I}_{\{k\Delta \in \mathbb{N}\}}$ is equal to 1 when $k\Delta$ corresponds to a premium payment date and 0 otherwise.

After that, we proceed as in the previous section. For the sake of clearness, in what follows we repeat the valuation steps with the proper changes.

1. Assume first that the insured is still alive at time $t = T - \Delta$, that the contract has not been previously surrendered and that the current value of the accumulated investments in the reference fund is $F_{T-\Delta}^i$ ($i = 1, 2, \dots, 2^{K-1}$). Then the policyholder has two alternatives:
 - (a) to surrender the contract, and in this case he(she) immediately receives the surrender value $R_{T-\Delta}^i \equiv g_{T-\Delta}(F_{T-\Delta}^i)$;
 - (b) to continue the contract, and in this case the beneficiary will receive, at time T , the (*stochastic*) benefit C_T^M if the insured dies during the last time interval or, alternatively, the (*stochastic*) benefit C_T^V if the insured survives the maturity date. Moreover, if $\Delta=1$ (i.e., if $K=T$), the policyholder immediately pays the premium P .

Hence the continuation value and the value of the whole contract are respectively given by

$$W_{T-\Delta}(P) = W_{T-\Delta}^i(P) \equiv \Delta q_{x+T-\Delta} \{e^{-r\Delta} [q f_T^M(F_T^{2i-1}) + (1-q) f_T^M(F_T^{2i})]\} \\ + \Delta p_{x+T-\Delta} \{e^{-r\Delta} [q f_T^V(F_T^{2i-1}) + (1-q) f_T^V(F_T^{2i})]\} - P \mathbb{I}_{\{T-\Delta \in \mathbb{N}\}},$$

$$i = 1, 2, \dots, 2^{K-1} \quad (24)$$

and

$$V_{T-\Delta}(P) = V_{T-\Delta}^i(P) \equiv \max \{R_{T-\Delta}^i, W_{T-\Delta}^i(P)\}, \quad i = 1, 2, \dots, 2^{K-1}. \quad (25)$$

2. Assume now that the insured is still alive at time $t = k\Delta$, $k = 0, 1, \dots, K - 2$, that the contract has not been previously surrendered and that the current value of the accumulated investments in the reference fund is $F_{k\Delta}^i$ ($i = 1, 2, \dots, 2^k$). Then the policyholder has two alternatives:
 - (a) to surrender the contract, and in this case he(she) immediately receives the surrender value $R_{k\Delta}^i \equiv g_{k\Delta}(F_{k\Delta}^i)$;

(b) to continue the contract, and in this case he(he) immediately pays the premium P , if $k\Delta$ is an integer. Moreover, the beneficiary will receive, at time $(k+1)\Delta$, the (*stochastic*) benefit $C_{(k+1)\Delta}^M$ if the insured dies between times $k\Delta$ and $(k+1)\Delta$ or, alternatively, the policyholder will be entitled to a contract with (*stochastic*) value, at time $(k+1)\Delta$, given by $V_{(k+1)\Delta}(P)$.

Hence the continuation value and the value of the whole contract are respectively given by

$$\begin{aligned} W_{k\Delta}(P) &= W_{k\Delta}^i(P) \equiv \Delta p_{x+k\Delta} \{e^{-r\Delta} [qV_{(k+1)\Delta}^{2i-1}(P) + (1-q)V_{(k+1)\Delta}^{2i}(P)]\} \\ &\quad + \Delta q_{x+k\Delta} \{e^{-r\Delta} [qf_{(k+1)\Delta}^M(F_{(k+1)\Delta}^{2i-1}) + (1-q)f_{(k+1)\Delta}^M(F_{(k+1)\Delta}^{2i})]\} \\ &\quad - P\mathbb{I}_{\{k\Delta \in \mathbb{N}\}}, \quad k = 0, 1, \dots, K-2 \quad \text{and} \quad i = 1, 2, \dots, 2^k \end{aligned} \quad (26)$$

and

$$V_{k\Delta}(P) = V_{k\Delta}^i(P) \equiv \begin{cases} W_0^1(P) & \text{if } k=0 \\ \max \{R_{k\Delta}^i, W_{k\Delta}^i(P)\} & \text{if } k=1, \dots, K-2 \text{ and } i=1, \dots, 2^k \end{cases} \quad (27)$$

Observe that Remark 1 of Section 2.3 is still valid, whatever the particular value of P on which $V_{k\Delta}$ and $W_{k\Delta}$ depend may be. The computational complexity of the above procedure is exponential, since the total nodes to be “visited” are now $1 + 2 + \dots + 2^{K-1} = 2^K - 1$.

As already said, this procedure supplies a time 0 value of the contract dependent on P , even in the exogenous case, say

$$V_0(P) = h(P). \quad (28)$$

Then the natural way to define a “fair” premium for the contract is to require that the time 0 value of the liabilities of the insurance company equals the time 0 value of the liabilities of the policyholder, i.e., that P satisfies the equation

$$h(P) = 0. \quad (29)$$

Once again, this premium is well defined if and only if the function h admits a unique zero. This always happens because

- i) h is a continuous function on all non-negative real numbers,
- ii) $h(0) > 0$,
- iii) $\lim_{P \rightarrow +\infty} h(P) = -\infty$,
- iv) h is strictly decreasing.

It is immediate to verify Properties i) and ii) and, recalling that the benefit, at death or maturity, and the surrender value are assumed to be independent of P (at least for the moment), it is also very easy to prove iii) and iv). In particular, the proof that h is strictly decreasing can be supplied by backward induction or, alternatively, by exploiting no-arbitrage arguments. Here we sketch the proof by induction, because a more general no-arbitrage proof will be presented in the next subsection, when we will deal with endogenous guarantees and surrender values.

Proof of iv)

Observe, first of all, that $W_{T-\Delta}$, given by relation (24), is constant (or strictly decreasing if $\Delta=1$), hence weakly decreasing. Then $V_{T-\Delta}$ is weakly decreasing as well (see relation (25)). Assume now that $V_{(k+1)\Delta}$ ($k = 0, 1, \dots, K-2$) is weakly decreasing, hence also $W_{k\Delta}$ is weakly decreasing, and strictly decreasing when $k\Delta$ is an integer (see relation (26)). This implies, once again, that $V_{k\Delta}$ is weakly decreasing (see relation (27)) and, in particular, $h = V_0 = W_0$ is strictly decreasing \square

The unique annual premium P can then be numerically computed by applying simple iterative methods. We conclude this subsection by observing that Remarks 2 and 3a) of Section 2.3 can be suitably adapted also to the annual premium case.

3.3 Endogenous guarantees and surrender values

Assume now that the benefit, at death or maturity, and the surrender value are endogenously specified as follows:

$$C_t^M = R_t = \max \left\{ F_t, P \sum_{j=0}^{n(t)} e^{\delta(t-j)} \right\}, \quad \delta \geq 0 \quad \text{and} \quad 0 < t < T \quad (30)$$

and

$$C_T^M = C_T^V = \max \left\{ F_T, P \sum_{j=0}^{T-1} e^{\delta(T-j)} \right\}, \quad \delta \geq 0, \quad (31)$$

where $n(t)$ is defined in relation (20). Note that the minimum guarantee at death, surrender or maturity is now given by the amount of the periodic premiums paid at the beginning of each year of contract with accrued interests at rate δ .

As in the exogenous case dealt with in the previous subsection, the time 0 value of the whole contract is a (modified) function of P , say

$$V_0(P) = \varphi(P), \quad (32)$$

and the contract is fairly priced if and only if the equation

$$\varphi(P) = 0 \quad (33)$$

has a unique solution.

Note that, while it is still immediate to verify that φ is a continuous function on all non-negative real numbers and that $\varphi(0) > 0$, now it is no longer guaranteed,

at least in general, that this function is strictly decreasing and that its limit, for $P \rightarrow +\infty$, is $-\infty$, as in the exogenous case. In fact, in this case the benefits and the surrender values are (weakly) increasing functions of P , and this could make the continuation values be increasing as well.

As in the single premium case, the following proposition provides necessary and sufficient conditions for the premium to be well defined:

Proposition 2

A necessary and sufficient condition for the existence and uniqueness of a premium P satisfying relation (33) is $\delta < r$.

Proof

Necessity:

Assume that there exists a premium P satisfying (33). Ab absurd, assume moreover that $\delta > r$. Then the policyholder could buy the contract and borrow P , at the risk-free rate r , at each premium payment date until the contract is still in force. At surrender, or death, or maturity, he(he) (or the beneficiary) would receive at least the amount of the premiums paid with accrued interests at rate δ , with which he(he) could pay off his(her) debt and realize (at least) a strictly positive interest gain. This is obviously impossible, since arbitrage opportunities are ruled out of the market, hence δ must be $\leq r$. However, if $\delta=r$, it is easy to verify that $\varphi(P)=0$ for any $P \geq D \sum_{j=0}^{T-1} u^{(T-j)/\Delta} / \sum_{j=0}^{T-1} e^{r(T-j)}$ because in these cases the guarantees are always effective (i.e., $P \sum_{j=0}^{n(k\Delta)} e^{r(k\Delta-j)} \geq F_{k\Delta}^i$ for any $k=1, 2, \dots, K$ and $i=1, 2, \dots, 2^k$) and the contract simply becomes a sequence of periodic investments in the riskless asset. Hence the uniqueness of the solution implies $\delta \neq r$.

Sufficiency:

As already observed, the function φ is continuous on all non-negative real numbers and, moreover, it is easy to verify, by backward induction, that our recursive procedure implies

$$\varphi(0) = D \sum_{j=0}^{T-1} e^{-rj} {}_j p_x > 0$$

(because this corresponds to the case of a free contract without guarantees) and that

$$\varphi(P) = P [e^{(\delta-r)\Delta} - 1] \quad \text{for any } P \geq D \frac{\sum_{j=0}^{T-1} u^{(T-j)/\Delta}}{\sum_{j=0}^{T-1} e^{\delta(T-j)}}$$

(because in these cases the guarantees become always effective). This last fact, along with the condition $\delta < r$, implies

$$\lim_{P \rightarrow +\infty} \varphi(P) = \lim_{P \rightarrow +\infty} P [e^{(\delta-r)\Delta} - 1] = -\infty.$$

Hence the existence of a zero for φ is guaranteed by the properties of continuous functions.

The uniqueness of the zero follows from the fact that φ is a strictly decreasing function. To prove this assume, ab absurd, that there exist two non-negative values

P_1 and P_2 such that $P_1 < P_2$ and $\varphi(P_1) \leq \varphi(P_2)$. Then the insurance company could

- a) assume a “short position” in a contract with annual premium P_2 and guarantees linked to P_2 . This means that it sells the contract to the policyholder who pays immediately the first premium P_2 and, moreover, will pay the same premium at the beginning of each year of contract until death, surrender or maturity, whichever comes first. At termination, the insurance company will pay the benefit (or the surrender value) with minimum guarantee given by the amount of premiums collected with accrued interests at rate δ . Since this contract is not necessarily fair (i.e., $\varphi(P_2)$ could be different from 0), the insurance company immediately “collects” also the (possibly negative) amount $\varphi(P_2)$, i.e., it pays $-\varphi(P_2)$ if $\varphi(P_2) < 0$ or receives $\varphi(P_2)$ if $\varphi(P_2) \geq 0$;
- b) assume a “long position” in an identical contract (on the same life) but with annual premium P_1 and guarantees linked to P_1 . This means that it buys the contract from a reinsurer, paying immediately the first premium P_1 and after the other premiums at the beginning of each year of contract until the contract is still in force. For this position the insurance company “pays” immediately also the (possibly negative) amount $\varphi(P_1)$;
- c) invest in the riskless asset the amounts $P_2 - P_1$ whenever a premium is paid, included that received with certainty at time 0.

This strategy would produce a non-negative inflow at time 0, given by $\varphi(P_2) - \varphi(P_1)$. Assume that the insurance company surrenders its reinsurance contract only if and when the policyholder surrenders his(her) contract. Then, at time of termination, say t ($\leq T$), the insurance company would

- a) pay the benefit (or the surrender value) $\max \left\{ F_t, P_2 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \right\}$ to the beneficiary (or to the policyholder) of the contract sold;
- b) receive $\max \left\{ F_t, P_1 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \right\}$ from the reinsurer;
- c) sell back the riskless assets, collecting the amount $(P_2 - P_1) \sum_{j=0}^{n(t)} e^{r(t-j)}$.

The final payoff of this strategy, given by

$$\begin{aligned} & \max \left\{ F_t, P_1 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \right\} - \max \left\{ F_t, P_2 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \right\} + (P_2 - P_1) \sum_{j=0}^{n(t)} e^{r(t-j)} \\ &= \begin{cases} (P_2 - P_1) \sum_{j=0}^{n(t)} [e^{r(t-j)} - e^{\delta(t-j)}] & \text{if } F_t \leq P_1 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \\ F_t - P_2 \sum_{j=0}^{n(t)} e^{\delta(t-j)} + (P_2 - P_1) \sum_{j=0}^{n(t)} e^{r(t-j)} & \text{if } P_1 \sum_{j=0}^{n(t)} e^{\delta(t-j)} < F_t < P_2 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \\ (P_2 - P_1) \sum_{j=0}^{n(t)} e^{r(t-j)} & \text{if } F_t \geq P_2 \sum_{j=0}^{n(t)} e^{\delta(t-j)} \end{cases} \end{aligned}$$

would be strictly positive whatever F_t may be, hence an arbitrage opportunity would exist¹³ \square

Simple iterative methods can then be employed in order to find the unique zero of φ .

3.4 Numerical results

To ascertain possible differences of the behaviour of the premiums for guarantees, at death or maturity, and for the surrender option in the annual premium case with respect to that observed in the single premium one, hereafter we discuss some numerical results. As in Section 2.5, we have fixed $D=100$, so that the annual premiums for the guarantees and for the surrender option are expressed as percentages of the amount deemed to be periodically invested in the reference fund. Moreover, also in this case we have computed, first of all, the annual premium for the European version of the contract, say P^E , and after the premium P for the whole contract. Then, denoting by G the annual premium for possible guarantees at death or maturity and by H that for the surrender option, once again we have set

$$G = P^E - D \quad \text{and} \quad H = P - P^E. \quad (34)$$

The mortality probabilities are the same used in the single premium case, as well as the parametrization defined in relation (17). The only difference is that now we have fixed $K=T$ (i.e., $\Delta=1$). We realize that this is a quite crude assumption, but the computational complexity of our valuation procedure imposes similar restrictions if one wants to get results in a reasonable time. Anyway we think that, in spite of this, the information one draws from such results can be very useful.

Starting with the case of exogenous guarantees we observe, first of all, that Remark 3b) of Section 2.3 cannot be extended to the periodic premium case. In fact, in this case, the surrender option can be worth even if the benefits at death and maturity are guaranteed while the surrender values are simply given by the current value of the accumulated investments in the reference fund F_t . Moreover, if the surrender values are deterministic amounts, the surrender option can be worth less than in the case in which the surrender values are instead given by the maximum between F_t and the same deterministic amounts. To verify this we have set, first of all

$$C_t^M = \max \left\{ F_t, D \sum_{j=0}^{n(t)} e^{g(t-j)} \right\}, \quad 0 < t \leq T \quad \text{and} \quad C_T^V = C_T^M,$$

where g is a minimum interest rate guaranteed, at death or maturity. Then we have denoted by H^F the annual premium for the surrender option corresponding to the case in which

$$R_t = F_t, \quad 0 < t < T,$$

¹³The same arguments could be exploited to prove that the function h defined in Section 3.2 for the exogenous case is strictly decreasing. The only difference is that the final payoff of the strategy would be simply given by the (strictly positive) amount $(P_2 - P_1) \sum_{j=0}^{n(t)} e^{r(t-j)}$ whatever the value of F_t may be because the payoff from both contracts (long and short) at termination would coincide.

by H^D that corresponding to the case

$$R_t = D \sum_{j=0}^{n(t)} e^{h(t-j)}, \quad 0 < t < T,$$

and, finally, H^M corresponds to the assumption

$$R_t^M = \max \left\{ F_t, D \sum_{j=0}^{n(t)} e^{h(t-j)} \right\}, \quad 0 < t < T,$$

where h is an interest rate guaranteed at surrender. The results obtained for different levels of g and h are reported in the following Table 8.

TABLE 8

The annual premium for the minimum guarantee at death or maturity (G) and for the surrender option (H^F, H^D, H^M) with respect to different levels of the guaranteed interest rates (g, h) and different specifications of the surrender values

h	$g = 0.00$		$g = 0.02$		$g = 0.04$	
	$G = 9.14$ $H^F = 0.33$		$G = 15.61$ $H^F = 0.39$		$G = 26.04$ $H^F = 0.42$	
0.00	$H^D = 7.78$	$H^M = 8.92$	$H^D = 3.26$	$H^M = 3.91$	$H^D = 0.20$	$H^M = 0.65$
0.02	$H^D = 12.81$	$H^M = 13.95$	$H^D = 6.96$	$H^M = 7.84$	$H^D = 1.00$	$H^M = 1.50$
0.04	$H^D = 20.22$	$H^M = 21.24$	$H^D = 13.93$	$H^M = 14.88$	$H^D = 4.61$	$H^M = 5.27$

Note: $D=100, x=40, T=20, \Delta=1, r=0.05, \sigma=0.3$.

From Table 8 one can see that actually H^F has always a strictly positive value, even if very low, and H^D is always strictly less than H^M . Moreover, comparing the results of Table 8 with those of Table 1, one can notice that the total charge for guarantees and surrender option in the annual premium contract is less enough than that required in the single premium one. In particular, the value of the surrender option H^M (or H^D) in Table 8 is less enough than H (in Table 1) when $g=0$ and $g=0.02$, while it is greater when $g=0.04$. In this last case, however, the higher cost of the surrender option is super-counterbalanced by the premium G , that in Table 8 is about 10% less than in Table 1. Note that the results reported in Tables 1 and 8 are not homogeneous, due to the different discretization step-size. In spite of this, we believe that the comparison between them can equally give some useful insights, although the numerical differences have not to be taken literally. To support this opinion, we have sampled some single premiums computed with the same (rough) discretization used in this section, and we have found that they are quite similar to those presented in Section 2.5.

The results of Table 8 make us hope that the cost for endogenizing the guarantees at death, maturity and surrender is acceptable, at least for reasonable values of the

parameters. To verify this, we let

$$C_t^M = R_t = \max \left\{ F_t, P \sum_{j=0}^{n(t)} e^{\delta(t-j)} \right\}, \quad 0 < t < T$$

and

$$C_T^V = C_T^M = \max \left\{ F_T, P \sum_{j=0}^{T-1} e^{\delta(T-j)} \right\},$$

and replicate what made in the single premium case. In the basic example ($x=40$, $T=20$, $\delta=2\%$, $r=5\%$, $\sigma=25\%$) we have obtained the exogenous premiums $G=11.66$ and $H=6.76$, while the total cost of endogenization is $E=30.37$ (hence $P = D + G + H + E = 148.79$). We note that the annual premium for endogenizing the minimum guarantees at death, maturity and surrender in this basic example remains still high, although the total charge $P - D$ is drastically reduced with respect to the single premium case (48.79 against 91.72). The results reported in the following Tables 9 to 11 show the behaviour of the annual premiums when the parameters r , δ , σ are moved as in Tables 5 to 7.

TABLE 9

The annual premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (P) for different levels of the riskless rate (r)

r	G	H	E	P
0.025	23.68	3.98	140.13	267.79
0.030	20.76	4.67	86.78	212.21
0.035	18.11	5.29	62.16	185.56
0.040	15.72	5.86	47.44	169.02
0.045	13.58	6.36	37.47	157.41
0.050	11.66	6.76	30.37	148.79
0.055	9.95	7.08	25.13	142.16
0.060	8.45	7.35	21.12	136.92
0.065	7.12	7.57	17.90	132.59
0.070	5.97	7.70	15.40	129.07
0.075	4.97	7.73	13.36	126.06
0.080	4.10	7.72	11.64	123.46
0.085	3.37	7.74	10.21	121.32
0.090	2.74	7.68	9.01	119.43
0.095	2.21	7.55	7.98	117.74
0.100	1.77	7.39	7.06	116.22

Note: $D=100$, $x=40$, $T=20$, $\Delta=1$, $\delta=0.02$, $\sigma=0.25$.

The behaviour of all the annual premiums reported in Table 9 with respect to the riskless rate r is the same as that observed in the single premium case. In particular, the comparison between Tables 9 and 5 shows that the charge for the minimum guarantees G required by annual premium contracts is quite enough less than that required by single premium ones, at least when r is not very far from δ . Moreover, the charge for the surrender option H in the annual premium case is a

bit higher than that required in the single premium one when r is close to δ , and viceversa when r is far from δ . Finally, the cost of endogenization E is decidedly less in the annual premium case, and results absolutely supportable when $r-\delta$ is high.

TABLE 10

The annual premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (P) for different levels of the minimum rate guaranteed (δ)

δ	G	H	E	P
0.000	6.18	7.54	15.89	129.61
0.005	7.27	7.45	18.34	133.06
0.010	8.53	7.29	21.36	137.18
0.015	9.98	7.05	25.29	142.32
0.020	11.66	6.76	30.37	148.79
0.025	13.58	6.37	37.37	157.32
0.030	15.77	5.87	47.12	168.76
0.035	18.28	5.25	61.97	185.50
0.040	21.13	4.53	86.82	212.48
0.045	24.36	3.71	140.96	269.03

Note: $D=100$, $x=40$, $T=20$, $\Delta=1$, $r=0.05$, $\sigma=0.25$.

The same comments concerning the comparison between the behaviour of the premiums in the annual premium case and in the single premium one with respect to r apply also when the parameter δ is concerned (see Tables 10 and 6). In fact, comparing Tables 9 and 10, one can see that all the annual premiums change very little with r and δ when the difference $r-\delta$ remains constant¹⁴.

TABLE 11

The annual premiums for the guarantees at death or maturity (G) and for the surrender option (H) in the exogenous case, the total cost of endogenization (E) and the total premium (P) for different levels of the volatility parameter (σ)

σ	G	H	E	P
0.05	0.00	0.00	0.00	100.00
0.10	1.02	2.81	2.62	106.45
0.15	4.01	4.41	8.38	116.80
0.20	7.72	5.64	17.60	130.96
0.25	11.66	6.76	30.37	148.79
0.30	15.61	7.84	47.04	170.49
0.35	19.47	8.88	67.82	196.17
0.40	23.19	9.92	93.14	226.25
0.45	26.72	10.94	123.27	260.93
0.50	30.06	11.94	158.26	300.26

Note: $D=100$, $x=40$, $T=20$, $\Delta=1$, $r=0.05$, $\delta=0.02$.

¹⁴Recall that the single premiums did not change at all.

The premiums' behaviour with respect to σ is the same as in the single premium case and, once again, the endogenization is very cheaper when the contract is paid by annual premiums (see Tables 11 and 7). Note that, when $\sigma=5\%$, the guarantees at death or maturity and the surrender option are completely valueless, hence the exogenous and the endogenous annual premium contracts coincide.

Summing up, all the numerical results concerning annual premium contracts confirm the same behaviour with respect to the key-parameters of the model observed for single premium ones, as expected, but the total charge required for guarantees, surrender option and endogenization, expressed as a percentage of the amount deemed to be invested in the reference fund, is always lower when the contract is paid by annual premiums. In particular, this reduction is mainly ascribed to the cost of endogenization, that results much lower. Although the numerical comparison may also be affected by the different discretization step-size, this cost reduction is not at all unexpected, because the periodic premium feature reduces the average duration of the policyholder's investments.

4 Conclusions

We have proposed a model for pricing a unit-linked life insurance policy, of the endowment type, embedding a surrender option. The valuation has been performed by merging the contract into a contingent-claims framework, characterized by perfectly competitive and frictionless markets, free of arbitrage opportunities. The contract considered is very general and allows, in particular, for minimum guarantees at death, maturity or surrender. Moreover, the (possible) guarantees can be exogenously given, i.e., independent of premiums, as well as endogenously determined. We have analysed both single and annual premium contracts. The numerical results have shown that annual premium contracts, although reproducing the same qualitative behaviour as single premium ones, can be very cheaper, specially when the endogenization of guarantees at death, maturity and surrender is taken into account.

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