Prospective Lifetables: Life Insurance Pricing and Hedging with Dynamic Mortality

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Abstract

In life insurance, actuaries have traditionally calculated premiums and reserves using a deterministic mortality intensity, which is a function of the age of the insured only. Over the course of the 20th century, the population of the industrialized world underwent a major mortality transition, with a dramatic decline in mortality rates. The mortality decline has been dominated by two major trends: a reduction in mortality due to infectious diseases affecting mainly young ages, and a decrease in mortality at old ages. These mortality improvements have to be taken into account to price long-term life insurance products and to analyse the sustainability of social security systems. In this paper, we argue that pricing and reserving for pension and life insurance products requires dynamic (or prospective) lifetables. We briefly review classic and recent projection methods and adopt a Poisson log-bilinear approach to estimate Portuguese Prospective Lifetables. The advantages of using dynamic lifetables are twofold. Firstly, it provides more realistic premiums and reserves, and secondly, it quantifies the risk of the insurance companies associated with the underlying longevity risks. Finally, we discuss possible ways of transferring the systematic mortality risk to other parties.

1 Introduction and motivation

It is well documented that human mortality globally declined during the course of the 20th century. Mortality improvements are naturally viewed as a positive change for individuals and as a substantial social achievement of developed countries. The mortality decline has been dominated by two major trends: a reduction in mortality due to infectious diseases affecting mainly young ages, and a decrease

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in mortality at old ages. Effectively, based on available demographic databases, we can conclude that human life span shows no sign of approaching a fixed limit imposed by biology. Rather, historical trends show that both average and the maximum life span have increased gradually during the 20th century.

All of this poses a serious challenge for the planning of public retirement systems, the long term risk management of supplemental pension plans as well as for the pricing and reserving for life insurance companies. To be more precise, human longevity trends affect not only old-age pensions but all components of social security systems, namely health care costs and disability and survivorship benefits. Likewise, other insurance products sold by private companies providing some sort of "living benefits" are affected by these developments in longevity (e.g. post-retirement health care protection).

Mortality improvements have an obvious impact on pricing and reserving for any kind of long-term living benefits, particularly on annuities. The calculation of expected present values requires an appropriate mortality projection in order to avoid significant underestimation of future costs.

In order to protect the company from mortality improvements, actuaries have different solutions, among them to resort to projected (dynamic or prospective) lifetables, i.e., lifetables including a forecast of future trends of mortality instead of static lifetables. Static lifetables are obtained using data collected during a specific period (1 to 4 years) whereas dynamic lifetables incorporate mortality projections. To illustrate the problems with this approach, consider a female individual born in 2006. Her mother is 30-year-old and her grand-mother 60. To estimate the life expectancy of the newborn, the death probability at age 30 will be her mother's one and at age 60 her grand-mother's one, observed in 2006. This means that in a situation where longevity is increasing, static lifetables underestimate lifelengths and thus premiums relating to life insurance contracts. Conversely, dynamic lifetables will project mortality into the future accounting for longevity improvements

The literature on the construction of projected lifetables is vast and growing.[1] The classical approach is to fit an appropriate parametric function (e.g. Makeham model) to each calendar year data, and then treat parameter estimates as independent time series, extrapolating their behaviour to the future on order to provide the actuary with projected lifetables. Despite simple, this approach has serious limitations. In the first place, this approach strongly relies on the ap-

[1] A detailed review of mortality projection methods can be found in Tuljapurkar and Boe (1998), Pitacco (2004), Wong-Fupuy and Haberman (2004) and Bravo (2007).

propriateness of the parametric function adopted. Secondly, parameter estimates are very unstable a feature that undermines the reliability of univariate extrapolations. Thirdly, the time series for parameter estimates are not independent and often robustly correlated. Although applying multivariate time series methods for the parameter estimates is theoretically possible, this will complicate the approach and introduce new problems.

Lee and Carter (1992) developed a simple model for describing the long term trends in mortality as a function of a simple time index. The method models the logarithm of a time series of age-specific death rates as the sum of an age-specific component that is independent of time and a second component, expressed as a product of a time-varying parameter denoting the general level of mortality, and an age-specific component that signals the sensitiveness of mortality rates at each age fluctuate when the general level of mortality changes. The model is fitted to data, and the resulting time-varying parameter estimates are then modelled and forecasted using standard Box-Jenkins time series methods. Finally, from this forecast of the general level of mortality, the projected age-specific death rates are derived using the estimated age-specific parameters.

Recently, Brouhns et al. (2002a,b) and Renshaw and Haberman (2003a,b) adopted a Poisson log-bilinear regression model to build projected lifetables, an approach developed to prevent some limitations inherent to the Lee & Carter (1992) original methodology. Indeed, Lee & Carter assume that the errors are homoskedastic, an unrealistic assumption since the logarithm of the force of mortality is normally much more variable at older ages that at younger ages. On the other hand, since the estimation of the model relies on a Singular Value Decomposition (SVD) of the matrix of the log age-specific observed forces of mortality, a complete rectangular matrix of data actually needed. Moreover, for actuarial applications, the law of the number of deaths is very useful and in this sense, the adoption of a Poisson distribution for the number of death remedies some of the Lee & Carter drawbacks.

In this paper we adopt a the Poisson log-bilinear approach developed by Brouhns et al. (2002a,b) to estimate Portuguese general population prospective lifetables. The results are then compared with classical static lifetables to give and indication of the longevity risk faced by insurance companies.

Let us now describe the content of this paper. Section 2 describes the notation and the assumptions adopted throughout this paper. Section 3 resumes the basic features of the Poisson log-bilinear projection model suggested by Brouhns et al. (2002a,b). Section 4 presents the data used in this study and describes the major

mortality trends in Portugal during the 20th century. Section 5 examines the results of the application of the projection model to the Portuguese data. Section 6 concludes.

2 Notation, assumption and quantities of interest

2.1 Notation

The basic idea underlying projected lifetable methods is to analyse changes in mortality as a function of both age x and time t. Even though age and time are theoretically free to oscillate in the half-positive real line, we assume here that x and t are integer numbers. From now on, $\mu_x(t)$ will denote the force of mortality at age x during calendar year t. By the same reason, $q_x(t)$ represents the one-year death probability at age x in year t and $p_x(t) = 1 - q_x(t)$ is the corresponding survival probability. Let $D_{x,t}$ denote the number of deaths recorded at age x during year t, from an exposure-to-risk (i.e., the number of person years from which $D_{x,t}$ arise) $E_{x,t}$.

2.2 Piecewize constant forces of mortality

Consider the classic Lexis diagram, that is, a coordinate system that has calendar time as abscissa and age as coordinate. If we assume that both time scales are divided into yearly bands, the Lexis plane is partitioned into squared segments. In this paper, we assume that the age-specific forces of mortality are constant within bands of time and age, but authorized to change from one band to the next. Formally, given any integer age x and calendar year t, we assume that

$$\mu_{x+\xi}(t+\tau) = \mu_x(t) \text{ for any } 0 \leq \xi, \tau < 1 \tag{1}$$

In other words, assumption (1) means that mortality rates are constant within each square of the Lexis diagram, but allowed to vary between squares. From (1) the calculation of the probability of an individual aged x in year t, $p_x(t)$, and of the corresponding death probability $q_x(t) = 1 - p_x(t)$ simplifies to:

$$p_x(t) = \exp(-\mu_x(t)) = 1 - p_x(t) \tag{2}$$

2.3 Quantities of interest

Several markers are regularly used by demographers to measure the evolution of mortality, namely life expectancies, variance of residual lifetime, median lifetime

or the entropy of a lifetable. Let $e_x(t)$ denote the life expectancy of an x-aged individual in year t, i.e., the average number of years he is expected to survive. This means we expect this individual will die in year $t+e_x(t)$ then aged $x+e_x(t)$. Contrary to classic static lifetables, the use of projected lifetables allows us to estimate the "true" expected residual lifetime of an individual. The appropriate formula for $e_x(t)$ is given by

$$\begin{aligned} e_x(t) &= \sum_{k\geq 0}\left\{\prod_{j=0}^{k} p_{x+j}(t+j)\right\} \\ &= \frac{1-\exp(-\mu_x(t))}{\mu_x(t)} \\ &\quad + \sum_{k\geq 1}\left\{\prod_{j=0}^{k-1}\exp\left(-\mu_{x+j}(t+j)\right)\right\}\frac{1-\exp\left(-\mu_{x+k}(t+k)\right)}{\mu_{x+k}(t+k)} \end{aligned} \tag{3}$$

The actual computation of $e_x(t)$ requires the knowledge of $\mu_\xi(\tau)$ (or $p_\xi(\tau)$) for $x \leq \xi \leq \omega$ and $t \leq \tau \leq t+\omega-x$, where ω denotes the ultimate (maximum) age. Since these survival probabilities are knot known at time t, they have to be estimated using extrapolation methods based on past trends. The next section gives an example of how this can be done in practice.

For life insurance companies and annuity providers, the net single premium of an immediate life annuity sold to an x-aged individual in year t, $a_x(t)$, is of special interest. The appropriate formula for $a_x(t)$ is given by

$$a_x(t) = \sum_{k\geq 0}\left\{\prod_{j=0}^{k} p_{x+j}(t+j)\right\}\upsilon^{k+1} \tag{4}$$

where $\upsilon = (1+i)^{-1}$ is the yearly discount factor. As can be seen, mortality projections and projected survival probabilities are particularly important to price correctly annuity and other life insurance contracts.

3 Mortality projection method

3.1 Poisson log-bilinear model

Following Brouhns et al. (2002a), we adopt a Poisson log-bilinear approach and consider that

$$D_{x,t} \sim \mathcal{P}oisson\left(\mu_x(t)\, E_{x,t}\right) \tag{5}$$

with

$$\mu_x(t) = \exp(\alpha_x + \beta_x \kappa_t) \tag{6}$$

where the parameters α_x, β_x and κ_t have to be constrained by

$$\sum_{t=t_{\min}}^{t_{\max}} \kappa_t = 0 \quad \text{and} \quad \sum_{x=x_{\min}}^{x_{\max}} \beta_x = 1 \tag{7}$$

in order to ensure model identification.

Similar to Lee and Carter (1992), the model assumes that the force of mortality has a log-bilinear structure, that is, $\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t$. Additionally, the expected number of death is eay to calculate and given by $\lambda_{x,t} = \mathbb{E}(D_{x,t}) = E_{x,t} \exp(\alpha_x + \beta_x \kappa_t)$. The meaning of parameters α_x, β_x and κ_t is fundamentally the same as in the traditional Lee-Carter model, that is,

$\exp(\alpha_x)$: corresponds to the general shape of mortality across age or, more rigorously, to the geometric mean of $\mu_x(t)$ in the observation period.

κ_t : denotes the general time trends in mortality

β_x : expresses the sensitivity of the logarithm of the force of mortality at age x to variations in the parameter κ_t, i.e., it basically determines the speed of reaction of mortality rates at each age in response to changes in general time trends.

3.2 Estimation of the parameters

One of the main advantages of the Poisson log-bilinear model over the Lee and Carter (1992) model is that specification (5) allows us to use maximum-likelihood methods to estimate the parameters instead of resorting to SVD methods. Moreover, since model (5) doesn't requires a SVD of the matrix $\ln \mu_x(t)$ we don't need a complete rectangular matrix of data anymore.

Formally, we estimate the parameters α_x, β_x and κ_t by maximizing the log-

likelihood derived from model (5)-(6), which is given by

$$\begin{aligned}
\ln \mathcal{V}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) &= \ln \left\{ \prod_{t=t_{\min}}^{t_{\max}} \prod_{x=x_{\min}}^{x_{\max}} \left(\frac{\lambda_{x,t}^{D_{x,t}} \exp(-\lambda_{x,t})}{(D_{x,t})!} \right) \right\} \\
&= \sum_{t=t_{\min}}^{t_{\max}} \sum_{x=x_{\min}}^{x_{\max}} \{ D_{x,t} \ln \lambda_{x,t} - \lambda_{x,t} - \ln[(D_{x,t})!] \} \\
&= \sum_{t=t_{\min}}^{t_{\max}} \sum_{x=x_{\min}}^{x_{\max}} \{ D_{x,t} \ln E_{x,t} + D_{x,t}(\alpha_x + \beta_x \kappa_t) \\
&\quad - E_{x,t} \exp(\alpha_x + \beta_x \kappa_t) - \ln[(D_{x,t})!] \} \\
&= \sum_{t=t_{\min}}^{t_{\max}} \sum_{x=x_{\min}}^{x_{\max}} \{ D_{x,t}(\alpha_x + \beta_x \kappa_t) - E_{x,t} \exp(\alpha_x + \beta_x \kappa_t) \} + c
\end{aligned} \tag{8}$$

where $\boldsymbol{\alpha} = (\alpha_{x_{\min}}, \ldots, \alpha_{x_{\max}})$, $\boldsymbol{\beta} = \left(\beta_{x_{\min}}, \ldots, \beta_{x_{\max}}\right)$, $\boldsymbol{\kappa} = (\kappa_{x_{\min}}, \ldots, \kappa_{x_{\max}})$ and c is a constant.

The presence of the bilinear term $\beta_x \kappa_t$ makes it impossible to estimate the model using standard statistical packages that include Poisson regression. Because of this, we resort to an iterative method for estimating log-linear models with bilinear terms proposed by Goodman (1979). The algorithm, which is essentially a Newton-Raphson standard method, states that in iteration $\upsilon + 1$, a single set of parameters is updated fixing the other parameters at their current estimates according to the following updating scheme

$$\hat{\theta}_j^{(\upsilon+1)} = \hat{\theta}_j^{(\upsilon)} - \frac{\partial \mathcal{L}^{(\upsilon)} / \partial \theta_j}{\partial^2 \mathcal{L}^{(\upsilon)} / \partial \theta_j^2} \tag{9}$$

where $\mathcal{L}^{(\upsilon)} = \mathcal{L}^{(\upsilon)}(\hat{\theta}^{(\upsilon)})$. Recall that in our case we have three sets of parameters, corresponding to the α_x, β_x and κ_t terms.

The updating scheme is as follows: starting with a given initial vector $(\hat{\alpha}_x^{(0)} \hat{\beta}_x^{(0)} \hat{\kappa}_t^{(0)})$, then:

$$
\begin{aligned}
\hat{\alpha}_x^{(v+1)} &= \hat{\alpha}_x^{(v)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \left[d_{x,t} - E_{x,t} \exp\left(\hat{\alpha}_x^{(v)} + \hat{\beta}_x^{(v)} \hat{\kappa}_t^{(v)} \right) \right]}{- \sum_{t=t_{\min}}^{t_{\max}} \left[E_{x,t} \exp\left(\hat{\alpha}_x^{(v)} + \hat{\beta}_x^{(v)} \hat{\kappa}_t^{(v)} \right) \right]}, \\
& \hat{\beta}_x^{(v+1)} = \hat{\beta}_x^{(v)}, \ \hat{\kappa}_t^{(v+1)} = \hat{\kappa}_t^{(v)} \qquad (10) \\
\hat{\kappa}_t^{(v+2)} &= \hat{\kappa}_t^{(v+1)} - \frac{\sum_{x=x_{\min}}^{x_{\max}} \hat{\beta}_x^{(v+1)} \left[d_{x,t} - E_{x,t} \exp\left(\hat{\alpha}_x^{(v+1)} + \hat{\beta}_x^{(v+1)} \hat{\kappa}_t^{(v+1)} \right) \right]}{- \sum_{x=x_{\min}}^{x_{\max}} \left(\hat{\beta}_x^{(v+1)} \right)^2 \left[E_{x,t} \exp\left(\hat{\alpha}_x^{(v+1)} + \hat{\beta}_x^{(v+1)} \hat{\kappa}_t^{(v+1)} \right) \right]}, \\
& \hat{\alpha}_x^{(v+2)} = \hat{\alpha}_x^{(v+1)}, \ \hat{\beta}_x^{(v+2)} = \hat{\beta}_x^{(v+1)} \\
\hat{\beta}_x^{(v+3)} &= \hat{\beta}_x^{(v+2)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \hat{\kappa}_t^{(v+1)} \left[d_{x,t} - E_{x,t} \exp\left(\hat{\alpha}_x^{(v+2)} + \hat{\beta}_x^{(v+2)} \hat{\kappa}_t^{(v+2)} \right) \right]}{- \sum_{t=t_{\min}}^{t_{\max}} \left(\hat{\kappa}_t^{(v+2)} \right)^2 \left[E_{x,t} \exp\left(\hat{\alpha}_x^{(v+2)} + \hat{\beta}_x^{(v+2)} \hat{\kappa}_t^{(v+2)} \right) \right]}, \\
& \hat{\alpha}_x^{(v+3)} = \hat{\alpha}_x^{(v+2)}, \ \hat{\kappa}_t^{(v+3)} = \hat{\kappa}_t^{(v+2)}
\end{aligned}
$$

We use as a criterion to stop the iterative procedure a very small increase of the log-likelihood function.

The maximum-likelihood estimations of the parameters generated by (10) do not match the identification constraints (7), and have thus to be adapted. This is guaranteed by changing the parameterization in the following manner:

$$
\kappa_t^* = (\hat{\kappa}_t - \bar{\kappa}) K \quad \text{and} \quad \beta_x^* = \frac{\hat{\beta}_x}{\sum_{x=x_{\min}}^{x_{\max}} \hat{\beta}_x} \qquad (11)
$$

where $\bar{\kappa}$ denotes average value for $\hat{\kappa}_t$, i.e.

$$
\bar{\kappa} = \frac{1}{t_{\max} - t_{\min} + 1} \sum_{t=t_{\min}}^{t_{\max}} \hat{\kappa}_t
$$

and where K is given by

$$
K = \sum_{x=x_{\min}}^{x_{\max}} \hat{\beta}_x
$$

from which we finally calculate

$$
\alpha_x^* = \hat{\alpha}_x + \hat{\beta}_x \bar{\kappa} \qquad (12)
$$

The new estimates α_x^*, β_x^* and κ_t^* fulfill the constraints (7) and provide the same $\hat{D}_{x,t}$ since $\hat{\alpha}_x + \hat{\beta}_x\hat{\kappa}_t = \alpha_x^* + \beta_x^*\kappa_t^*$. Note also that differentiating the log-likelihood function with respect to α_x yields the equality

$$\sum_t D_{x,t} = \sum_t \hat{D}_{x,t} = \sum_t E_{x,t} \exp\left(\hat{\alpha}_x + \hat{\beta}_x\hat{\kappa}_t\right)$$

This means that the estimated κ_t's are such that the resulting death rates applied to the actual risk exposure produce the total number of deaths actually observed in the data for each age x.

3.3 Modelling the time-factor

In the Poisson log-bilinear methodology, the time factor κ_t is intrinsically viewed as stochastic process. In this sense, standard Box-Jenkins techniques are used to estimate and forecast κ_t within an ARIMA(p, d, q) time series model. Recall that the model takes the general form

$$(1 - B)^d \kappa_t = \mu + \frac{\Theta_q(B)\,\epsilon_t}{\Phi_q(B)} \tag{13}$$

where B is the delay operator (i.e., $B(\kappa_t) = \kappa_{t-1}$, $B^2(\kappa_t) = \kappa_{t-2}, \ldots$), $1 - B$ is the difference operator (i.e., $(1-B)\kappa_t = \kappa_t - \kappa_{t-1}$, $(1-B)^2\kappa_t = \kappa_t - 2\kappa_{t-1} + \kappa_{t-2}, \ldots$), $\Theta_q(B)$ is the Moving Average polynomial, with coefficients $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_q)$, $\Phi_q(B)$ is the Autoregressive polynomial, with coefficients $\boldsymbol{\phi} = \left(\phi_1, \phi_2, \ldots, \phi_p\right)$, and ϵ_t is white noise with variance σ_ϵ^2.

The method used to derive estimates for the ARIMA parameters μ, $\boldsymbol{\theta}$, $\boldsymbol{\phi}$ and σ_ϵ is conditional least squares. From these, forecasted values of the time parameter, denoted by κ_t^*, are derived. Finally, the parameter estimates of the Poisson model and the forecasts κ_t^* can be inserted in (6) to obtain age-specific mortality rates, prospective lifetables, life expectancies, annuities single premiums and other related markers. In the following we apply the Poisson modelling to Portugal's general population data in order to derive prospective lifetables.

4 Building prospective lifetables for Portugal

4.1 Data

The model used in this paper is fitted to the matrix of crude Portuguese death rates, from year 1970 to 2004 and for ages 0 to 84. The data, discriminated by age and sex, refers to the entire Portuguese population and has been supplied

by the Portuguese National Institute of Statistics (INE - Instituto Nacional de Estatística). The database for this study comprises two elements: the observed number of death $d_{x,t}$ given by age and year of death, and the observed population size $l_{x,t}$ at December 31 of each year. We follow the INE definition of population at risk using the population counts at the beginning and at the end of a year and take migration into account.

Figures 1 and 2 give us a first indication of mortality trends in Portugal during this period. Two trends dominated the mortality decline: (i) a reduction in mortality due to infectious diseases affecting mainly young ages, (ii) decreasing mortality at old ages.

4.2 Parameter estimation

We apply the Poisson modelling to the Portuguese data presented above. The Poisson parameters α_x, β_x and κ_t implicated in (6) are estimated by maximum-likelihood methods using the iterative procedure described in Section 3.2. We started the updating scheme considering the following initial values $\hat{\alpha}_x^{(0)} = 0$, $\hat{\beta}_x^{(0)} = 1$, and $\hat{\kappa}_t^{(0)} = 0.1$. The criterion to stop the iterative procedure is a very small increase of the log-likelihood function (in our case we used 10^{-5}). The routine was implemented within the SAS package. Figure 3 plots the estimated α_x, β_x and κ_t.

We note that the $\hat{\alpha}_x$'s represent the average of the $\ln \hat{\mu}_x(t)$ across the time period. As expected, the average mortality rates are relatively high for newborn and childhood ages, then decrease rapidly towards their minimum (around age 12), increasing then in x, reflecting higher mortality at older ages. The only exception refers to the well know "mortality hump" around ages 20-25, more visible in the male population, a phenomena normally associated with accident or suicide mortality. We can see that young ages tend to be more affected by changes in the general time trends of mortality, probably due the evolution of medicine in reducing infantile and juvenile mortality. In effect, the $\hat{\beta}_x$'s decrease with age, except for the mortality hump phenomena, but remain positive for all ages. Note also that the sensitiveness of the male population to variations in parameter κ_t tends to be grater than that of the female population, which has a more stable pattern. Finally, we can see that the $\hat{\kappa}_t$'s exhibit a clear decreasing trend (approximately linear). This reveals the significant improvements of mortality at all ages both for men and women in the last 35 years.

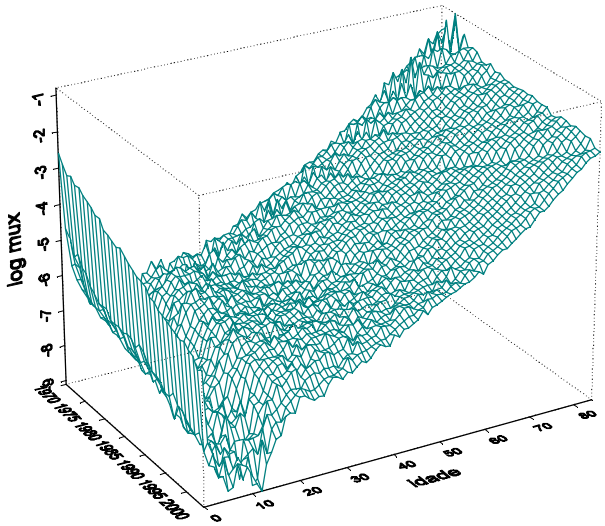

Figure 1: Crude mortality rates for the period 1970-2004, males

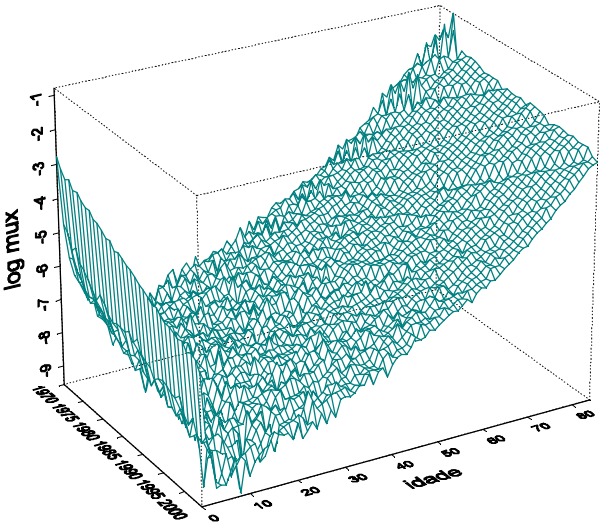

Figure 2: Crude mortality rates for the period 1970-2004, females

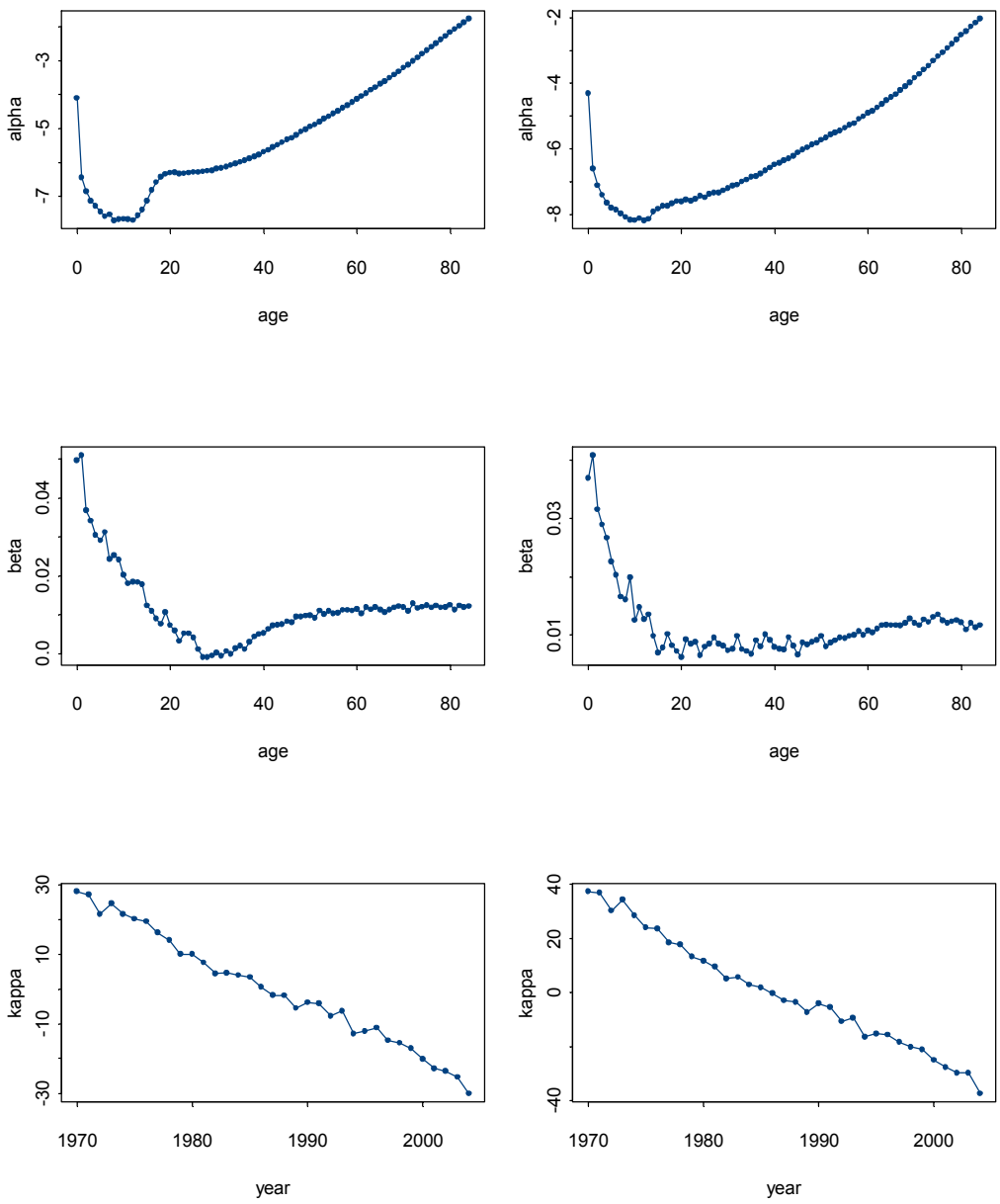

Figure 3: Estimations of α_x, β_x and κ_t for men (left panels) and women (right panels).

4.3 Extrapolating time trends

Let $\{\hat{\kappa}_t, t = t_{\min}, \ldots, t_{\max}\}$ denote a realization of the finite chronologic time series $\mathcal{K} = \{\kappa_t, t \in \mathbb{N}\}$. Following the work of Lee and Carter (1992) and Brouhn et al. (2002a,b), we use standard Box-Jenkins methodology to identify, estimate and extrapolate the appropriate ARIMA (p, d, q) time series model for the male and female time indexes κ_t.

A good model for the male population is ARIMA$(0, 1, 1)$, which is a moving average (MA(1)) model

$$(1 - B)\,\kappa_t^m = \rho^m + \theta^m \varepsilon_{t-1}^m + \varepsilon_t^m \tag{14}$$

whereas for women the ARIMA(1,1,0) autoregressive model was identified as a good candidate

$$(1 - B)\,\kappa_t^w = \rho^w + \phi^w \kappa_{t-1}^w + \varepsilon_t^w \tag{15}$$

where ε_t^m and ε_t^w are white noise error terms with variance σ_m^2 and σ_w^2, respectively. The estimated parameters for the ARIMA (p, d, q) models (14) and (15) are given in Table 1. Note that all parameters are significant at a 5% significance level.

Sex	Parameter	Estimate	Std error	$t-$value	$p-$value
Men	ρ^m	-1.64623	0.11663	-14.11	<.0001
	θ^m	0.64315	0.14831	4.34	0.0001
	σ_m	1.800992			
Women	ρ^w	-2.14802	0.23969	-8.96	<.0001
	ϕ^w	-0.63606	0.15145	-4.20	0.0002
	σ_w	2.263249			

Table 1: Estimation of the parameters of the ARIMA(p,d,q) models

In Figure 4 we show the estimated values of κ_t together with the κ_t^* projected and the corresponding 95% confidence interval forecasts.

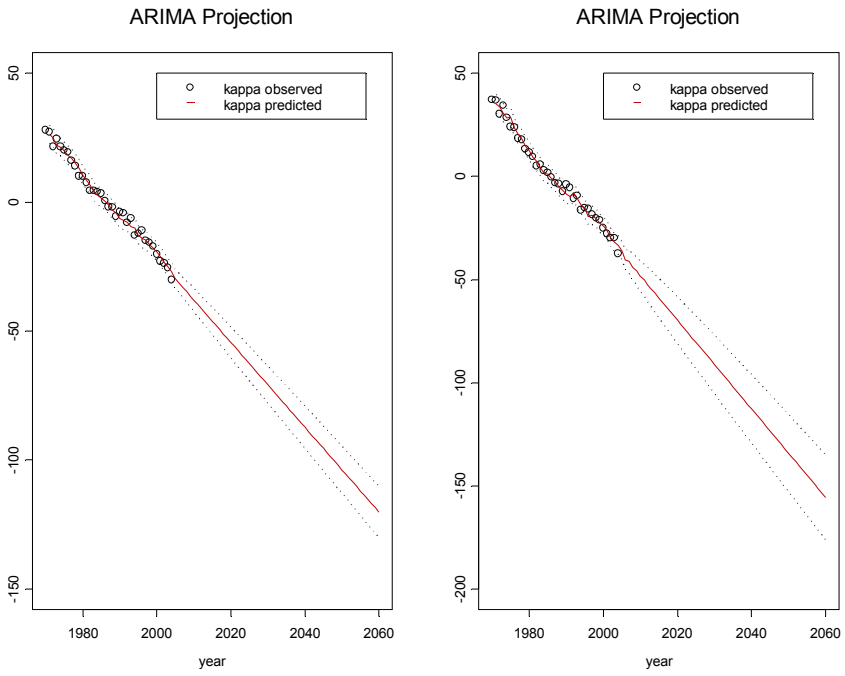

Figure 4: Estimated and projected values of κ_t with their 95% confidence intervals for males (left panel) and females (right panel)

Given the forecasted values of κ_t $\left\{\hat{\kappa}^*_{2004+s} : s = 1, 2, \ldots\right\}$, the reconstituted sex-specific forces of mortality are given by

$$\hat{\mu}_x\,(2004+s) = \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}^*_{2004+s}), \quad s = 1, 2, \ldots \tag{16}$$

and then used to generate sex-specific life expectancies and life annuities.

4.4 Completion of lifetables

According to the United Nations, it is estimated that in 2001 72 million of the 6.1 billion inhabitants of the world were 80 year or older. In the developing world, the population of the oldest-old (e.g., those 80 years and older) still represents a small fraction of the world's population but it is the fastest growing segment of the population. In addition, because life expectancy will continue to increase, not only we should expect to have an increasing number of people surviving to very old ages, but also anticipate that the deaths of the oldest-old will account for an increasing proportion of all deaths in a given population. In view of this, it is important to have detailed information about the age structure of the oldest-

old and about the behaviour of mortality at these ages. Unfortunately, in most countries reliable data on both the age distribution of population at risk and death counts of the oldest-old is not yet available. This is also our case since Portuguese statistics do not provide an age breakdown for the group aged 85 and over. This poses a serious problem when it comes to complete lifetables.

Because of this, a number of research papers has addressed the issue of projecting mortality for the oldest-old (see, e.g., Buettner (2002)). In this paper we adopt the method proposed by Denuit and Goderniaux. (2005) to extrapolate mortality rates at very old ages. The method is a two step method: first, a quadratic function is fitted to age-specific estimated mortality rates in a given age-band; second, the estimated function is used to extrapolated mortality rates up to a pre-determined maximum age. Formally, the following log-quadratic model is fitted by weighted least-squares

$$\ln \hat{q}_x(t) = a(t) + b(t)\, x + c(t)\, x^2 + \epsilon_x(t), \qquad 65 \leq x \leq 84 \tag{17}$$

to age-specific mortality rates observed at older ages (in our case $65 \leq x \leq 84$), where $\epsilon_x(t) \sim \mathcal{N}\left(0, \sigma^2(t)\right)$, with additional constraints

$$q_{120} = 1 \tag{18}$$

$$q'_{120} = 0 \tag{19}$$

where q'_x denotes the first derivative of q_x with respect to age x. Constraints (18) and (19) impose a concave configuration to the curve of mortality rates at old ages and the existence of a horizontal tangent at $x = 120$. We then use this function to extrapolated mortality rates up to age 120. Figures 5 and 6 show the final result of this procedure.

4.5 Mortality Projections

4.5.1 By chronologic year

Considering the prospective lifetables derived in the previous section, we can now analyse the evolution of mortality across time. Figure 7 represents the evolution of observed and estimated forces of mortality from 1970 to 2050 for both genders. In Figure 8 we can observe the evolution of observed and estimated mortality rates from 1970 to 2050 for both genders and some representative ages.

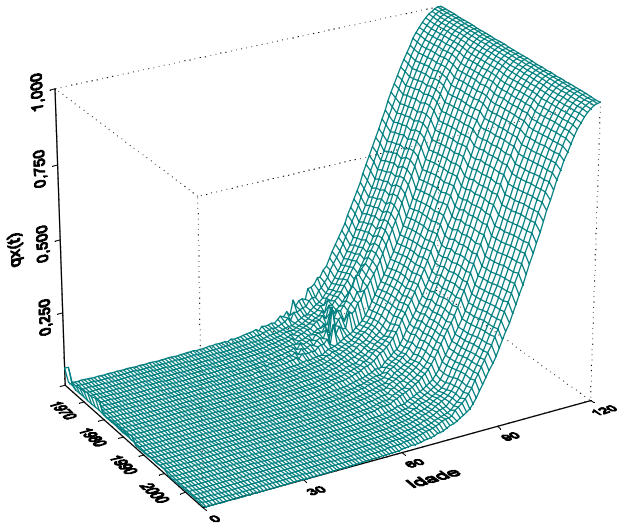

Figure 5: Mortality rates for closed lifetables, males

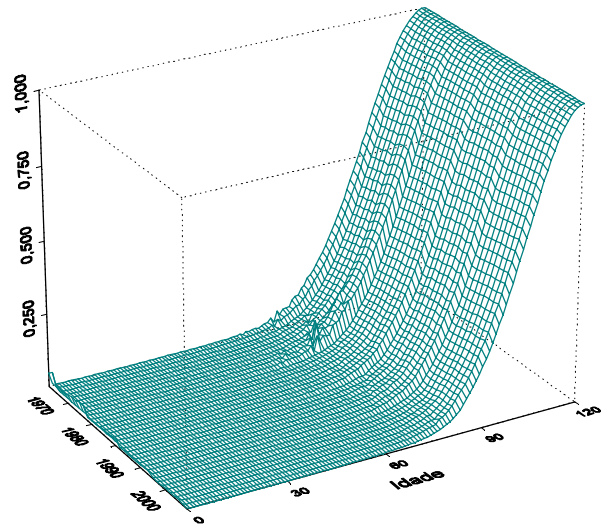

Figure 6: Mortality rates for closed lifetables, females

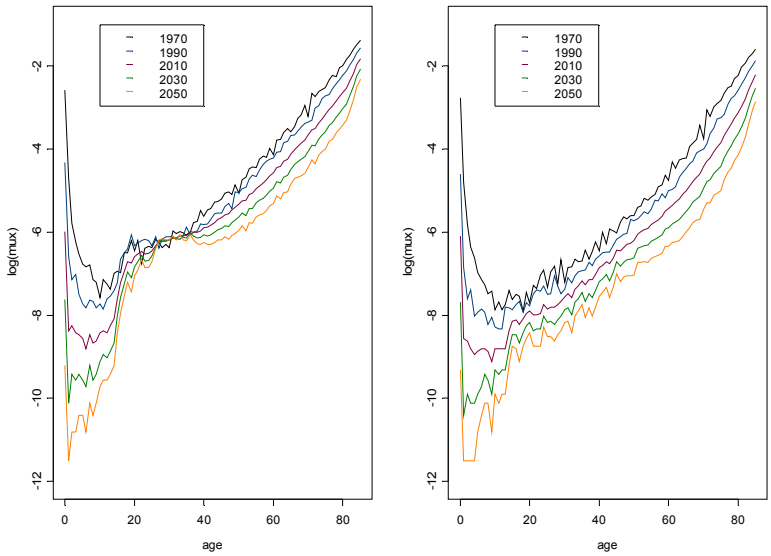

Figure 7: Evolution of $\mu_x(t)$ for men (left panel) and women (right panel)

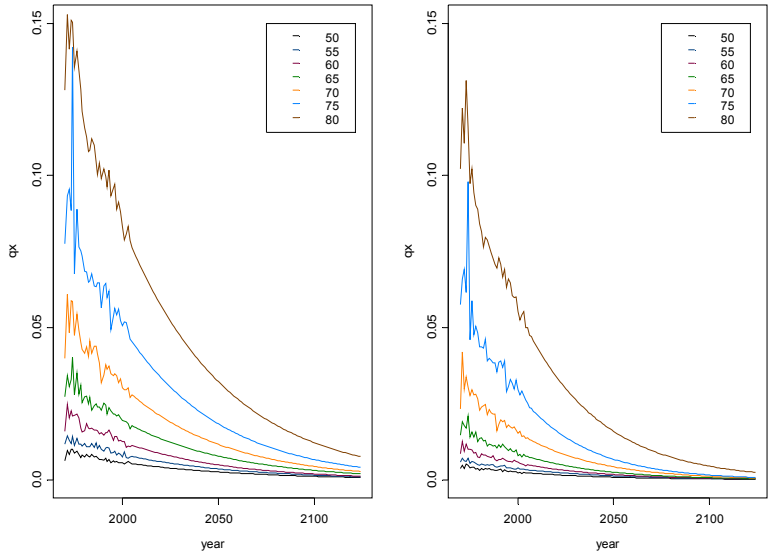

Figure 8: Evolution of q_x for some representative ages, from 1970 to 2124, for men (left) and women (right)

Overall, we can observe a clear and continuous decline in mortality throughout this period. It is also apparent that this mortality decline is more noticeable within younger ages. The mortality hump phenomenon is surprisingly persistent and tends to be more significant for the male population. In effect, we can observe a sort of mortality stagnation within this age-band. For older ages, we predict a decline in mortality rates.

4.5.2 By Cohort

Prospective lifetables provide us with new tools for the analysis of mortality trends, namely the possibility to investigate the evolution of mortality not only in terms of calendar time but also in terms of year of birth or cohort. In brief, by using prospective lifetables we switch from a transversal approach to a longitudinal (or diagonal) approach to mortality.

In Figure 9 we can observe the evolution of the force of mortality for some representative generations born between 1970 and 2004.

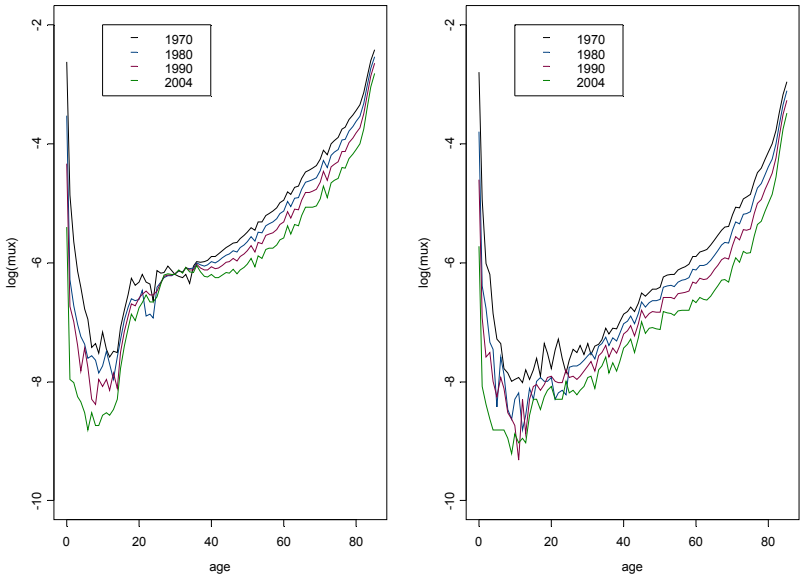

Figure 9: Evolution of the instantaneous force of mortality for some representative generations for men (left panel) and women (right panel)

We note that the main mortality features identified in the previous section within the transversal approach (decreasing mortality trends, mortality hump,...)

are again easily recognized within the cohort approach. It should be mentioned, however, that the evolution of mortality for successive generations seems to be more reliable and plausible when compared with that provided by the classic static approach.

In figure 10 we compare mortality rates obtained in both a transversal and diagonal approach for selected cohorts (and calendar years). We can observe that, in decreasing mortality environment, the predicted values within a diagonal approach are, as expected, lower than those estimated via a transversal approach. Note also that the differences in the projected values increase with the age of the individual and with the generation's year of birth. The only exception refers, once again, to the mortality hump phenomena, for which we project a stagnation (and even a slight increase) in mortality rates.

4.6 Life expectancy

In this section we analyse the evolution of life expectancy $e_x(t)$ in terms of calendar year $t = 1970, \ldots, 2004$ for some representative ages $x = 0$ and $x = 65$. In Section 2.3 we showed that within the transversal approach $e_x(t)$ is calculated on the basis of mortality rates observed (or estimated) in year t (i.e., using probabilities $q_{x+k}(t)$, $k = 0, 1, 2, \ldots$). For the contrary, within the diagonal approach $e_x(t)$ represents the "true" remaining lifetime for individuals aged x in year t, and is calculated on the basis of mortality rates projected for that generation (i.e., using probabilities $q_{x+k}(t+k)$, $k = 0, 1, 2, \ldots$). Table 2 summarizes the results obtained for the life expectancy calculated at birth and at age 65 for two selected calendar years. Column $\bar{\Delta}_y$ indicates the average annual gain (measured in days) in the life expectancy registered between 1970 and 2004.

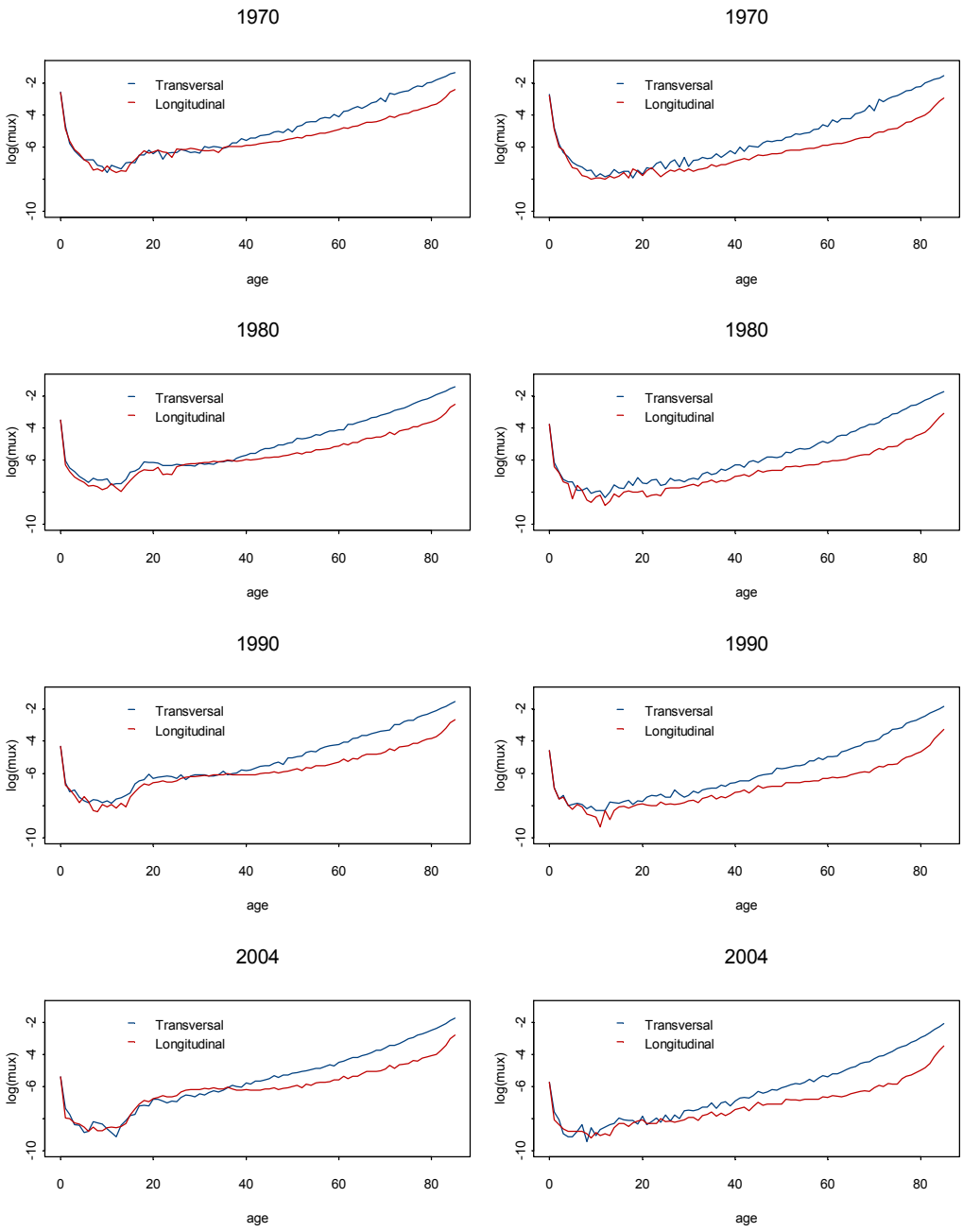

Figure 10: Transversal vs cohort approach, for selected calendar years, for men (left panel) and women (right panel)

Men	$e_0(t)$				$e_{65}(t)$			
t	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$
1970	71.99		63.21		13.16		12.21	
2004	83.30	117.9	74.55	118.3	16.91	39.2	15.84	38.0
Women	$e_0(t)$				$e_{65}(t)$			
t	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$
1970	80.37		69.32		16.24		14.53	
2004	90.63	107.0	81.05	122.4	20.87	48.3	19.22	49.0

Table 2: Evolution of life expectancy at birth and at age 65 calculated according to both a transversal and diagonal approach

The first noticeable aspect refers to the spectacular life expectancy gains observed during this period. In effect, when we can consider the transversal approach we observe that over this period life expectancy at birth increased, on an annual average, by approximately four months for both sexes (more precisely 118.3 and 122.4 days for men and women, respectively). These gains are slightly more moderate when considering the diagonal approach, particularly for the female population, with average annual gains amounting to 117.9 and 107.0 days for men and women, respectively. Similar conclusions may be stated when we examine the evolution of life expectancy at the age of 65.

The second main conclusion has do to with the significant difference between life expectancies estimated using the two approaches. In effect, when we use prospective lifetables we estimate that the "true" life expectancy at birth for an individual born in 2004 will be of 83.30 and 90.63 years for men and women, respectively, whereas the corresponding values estimated using the classic transversal approach are 74.55 and 81.05 years. In other words, when we project past trends observed in mortality to the future we conclude that adopting a transversal approach underestimates life expectancy at birth in 8.75 and 9.58 years for men and women, respectively. This apparently surprising conclusion highlights the importance of using prospective lifetables in life insurance and pension businesses. Actually, long-term calculations based on periodic lifetables are erroneous since they do not incorporate expected longevity improvements.

In Figure 11 we can see that the differentials between the values of $e_0(t)$ and $e_{65}(t)$ calculated according to the two methodologies considered are, for both sexes, relatively stable across the time period analysed.

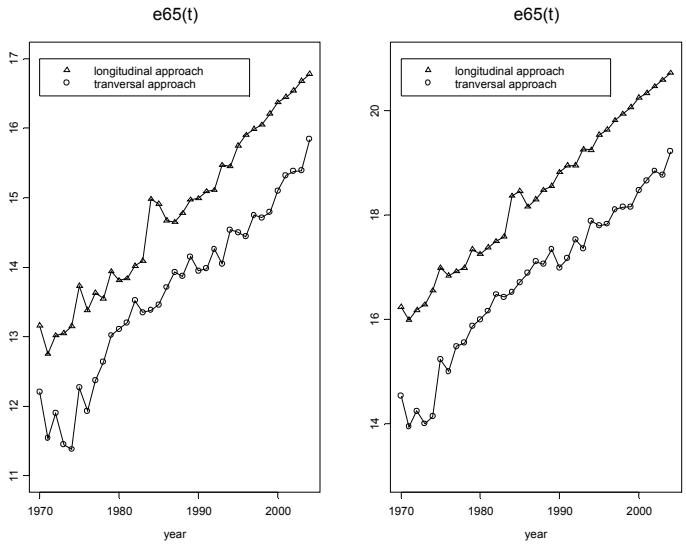

Figure 11: Life expectancy $e_x(t)$ calculated at $x = 0, 65$ for men (left panel) and women (right panel)

Finally, Figure 12 gives us a long term perspective of the evolution of $e_0(t)$ and $e_{65}(t)$ across the time period analysed.

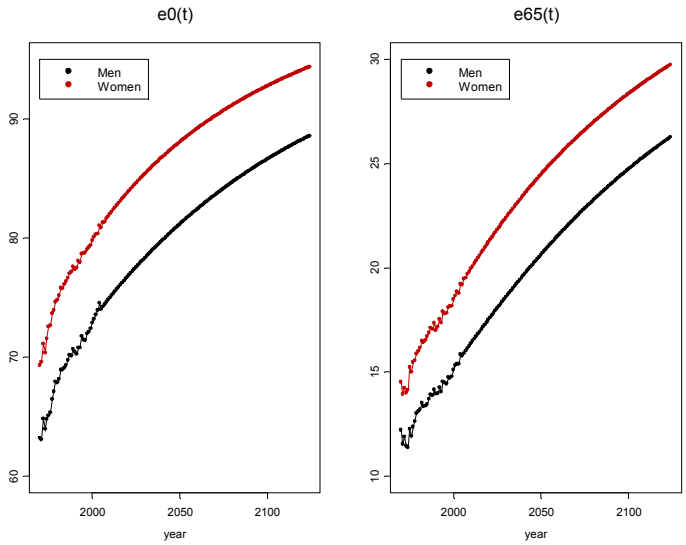

Figure 12: Projected life expectancy at birth and at age 65, calculated according a transversal approach

Our model estimates that life expectancy will continue to increase in the future in both sexes, although we expect longevity improvements to slow down.

4.7 Annuity prices

In this section we are interested in the evolution of the net single premium of an immediate life annuity sold to an x-aged individual in year t, $a_x(t)$ considered both a transversal and a diagonal approach. For simplicity of exposition, we assume a flat technical interest rate at 3%, i.e. $i = 3\%$. This means that we concentrate our analysis on the impact of longevity improvements on annuity prices. Given this, we examine the evolution of $a_x(t)$ for $x \in [0; 65]$ years.

Men				
t	$a_0(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_0(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	26.84		25.90	
2004	29.82	0.085	29.02	0.089
Women				
t	$a_0(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_0(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	28.18		27.06	
2004	30.72	0.073	29.89	0.081

Men				
t	$a_{65}(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_{65}(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	9.88		9.29	
2004	12.22	0.0668	11.64	0.0671
Women				
t	$a_{65}(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_{65}(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	11.86		10.85	
2004	14.56	0.077	13.70	0.081

Table 3: Evolution of $a_x(t)$ for $x = 0$ and $x = 65$

In Table 3 we can appreciate the underestimation of annuity prices resulting from classic transversal lifetables. For example, the net single premium of an immediate life annuity sold to a female individual aged 65 in year 2004, $a_{65}(2004)$,

will be 0.86$ higher $(14.56 - 13.70)$ or 6.3% when compared with that calculated using classic static lifetables. Values in column $\bar{\Delta}_y$(annual) indicate the average annual gains in $a_x(t)$ registered between 1970 and 2004.

5 Conclusion

To the knowledge of the authors, the present paper offers the first attempt to build prospective lifetables for the Portuguese population. In addition, we offer a first attempt to quantify the impact of longevity risk, that is, the risk arising from systematic deviations of observed mortality rates from their estimated values, on life annuity premiums computed on the basis of projected mortality rates.

The results obtained with the log-bilinear Poisson approach over a matrix of crude Portuguese death rates, from year 1970 to 2004 and for ages 0 to 84, clearly demonstrate that classic static lifetables tend to seriously underestimate the longevity prospects. Since mortality improvements have an obvious impact on pricing and reserving for any kind of long-term living benefits, particularly on annuities, we argue that the calculation of expected present values requires the use of prospective lifetables built using an appropriate mortality projection model.

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